Open Science Journal of Mathematics and Application
2014; 2(4): 37-43
Published online November 20, 2014 (http://www.openscienceonline.com/journal/osjma)

Bingmann-Lovejoy-Osburn's generating function in the overpartitions

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To cite this article

Fazlee Hossain, Sabuj Das, Haradhan Kumar Mohajan. Bingmann-Lovejoy-Osburn's Generating Function in the Overpartitions. *Open Science Journal of Mathematics and Application*. Vol. 2, No. 4, 2014, pp. 37-43.

Abstract

In 2009, Bingmann, Lovejoy and Osburn defined the generating function for $\overline{spt}(n)$. In 2012, Andrews, Garvan and Liang defined the $\overline{sptcrank}$ in terms of partition pairs. In this article the number of smallest parts in the overpartitions of n with smallest part not overlined is discussed, and the vector partitions and $\bar{S}$ -partitions with 4 components, each a partition with certain restrictions are also discussed. The generating function for $\overline{spt}(n)$, and the generating function for $M_{\bar{S}}(m,n)$ are shown with a result in terms of modulo 3. This paper shows how to prove the Theorem 1 in terms of $M_{\bar{S}}(m,n)$ with a numerical example, and shows how to prove the Theorem 2 with the help of $sptcrank$ in terms of partition pairs. In 2014, Garvan and Jennings-Shaffer are able to defined the $\overline{sptcrank}$ for marked overpartitions. This paper also shows another result with the help of 6 $\overline{SP}$-partition pairs of 3 and shows how to prove the Corollary with the help of 42 marked overpartitions of 6.

Keywords

Components, Congruent, Crank, Non-Negative, Overpartitions, Overlined, Weight

1. Introductions

In this paper we give some related definitions of $\overline{spt}\,(n)$, various product notations, vector partitions and $\bar{S}$ - partitions, $M_{\bar{S}}(m,n)$, $M_{\bar{S}}(m,t,n)$, marked partition and $\overline{sptcrank}$ for marked overpartitions. We discuss the generating function for $\overline{spt}\,(n)$ and prove the Corollary 1 with the help of generating function for $M_{\bar{S}}(m,n)$. We prove the Result 1 with the help of 8 vector partitions from $\bar{S}$ of 3. We prove the Theorem 1 with the help of various generating functions and establish the Corollary 2 with a special series $\bar{S}\,(z,x)$, when n =1, and prove the Theorem 2: This is;

$$\overline{spt}\,(n)=\sum_{\substack{\vec{\lambda}\in\overline{SP}\\ |\vec{\lambda}|=|\lambda_1|+|\lambda_2|=n}}1\,,$$

with a numerical example. We establish the Result 2 using the $\overline{crank}$ of partition pairs $\vec{\lambda}=(\lambda_1,\lambda_2)$ and analyze the Corollary 3 with the help of 42 marked overpartitions of 6.

2. Some Related Definitions

Here we introduce some definitions following [7].

$\overline{spt}(n)$ [4]: The number of smallest parts in the overpartitons of n with smallest part not overlined is denoted by $\overline{spt}\,(n)$ for example;

n	$\overline{spt}\,(n)$
1: $\dot{1}$	1
2: $\dot{2}$, $\dot{1}+\dot{1}$	3
3: $\dot{3}$, $2+\dot{1}$, $\bar{2}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}$	6
4: $\dot{4}$, $3+\dot{1}$, $\bar{3}+1$, $\dot{2}+\dot{2}$, $2+\dot{1}+\dot{1}$, $\bar{2}+\dot{1}+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}$	13
... ...	

Hence we get;

$$\overline{spt}(5) = 22,\ \overline{spt}(6) = 42, \ldots.$$

2.1. Product Notation

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)\ldots$$

$$(x^2;x^2)_\infty = (1-x^2)(1-x^4)\ldots$$

$$(x)_k = (1-x)(1-x^2)(1-x^3)\ldots(1-x^k)$$

$$(-x^5;x)_\infty = (1+x^5)(1+x^6)(1+x^7)\ldots$$

2.2. Vector Partitions and $\overline{S}$-Partitions

A vector partition can be done with 4 components each partition with certain restrictions [5]. Let $\vec{V} = D\times P\times P\times D$ where D denotes the set of all partitions into distinct parts, P denotes the set of all partitions. For a partition π, we let $s(\pi)$ denote the smallest part of π (with the convention that the empty partition has smallest part ∞), $\#(\pi)$ the number of parts in π, and $|\pi|$ the sum of the parts of π.

For $\vec{\pi} = (\pi_1,\pi_2,\pi_3,\pi_4)\in \vec{V}$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the crank $c(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$, the norm $\left|\vec{\pi}\right| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4|$.

We say $\vec{\pi}$ is a vector partition of n if $\vec{\pi} = n$. Let $\overline{S}$ denote the subset of $\overline{V}$ and it is given by $\bar{S} = \left\{(\pi_1,\pi_2,\pi_3,\pi_4)\in \vec{V}, 1\le s(\pi_1)<\infty, s(\pi_1)\le s(\pi_2), s(\pi_1)\le s(\pi_3), s(\pi_1)< s(\pi_4)\right\}$.

$M_{\overline{S}}(m,n)$: The number of vector partitions of n in $\overline{S}$ with crank m counted according to the weight ω is exactly $M_{\overline{S}}(m,n)$.

$M_{\overline{S}}(m,t,n)$: The number of vector partitions of n in $\overline{S}$ with crank congruent to m modulo t counted according to the weight ω is exactly $M_{\overline{S}}(m,t,n)$.

Marked Partition [1]: We define a marked partition as a pair (λ,k) where λ is a partition and k is an integer identifying one of its smallest parts i.e., k =1, 2, ... $\nu(\lambda)$, where $\nu(\lambda)$ is the number of smallest parts of λ.

$\overline{sptcrank}$ for Marked overpartitions [6]: We define a marked overpartitions of n as a pair (π, j) where π is an over partition of n in which the smallest parts is not overlined and j is an integer $1\le j\le \nu(\pi)$, where $\nu(\pi)$ is the number of smallest parts to π. It is clear that $\overline{spt}(n)$ = # of marked overpartitions (π, j) of n. For example, there are 3 marked overpartitions of 2 like;

(2,1),(1+1,1),(1+1,2), so that $\overline{spt}(2)$=3.

Again there are 6 marked overpartitions of 3 like;

(3,1), (2+1,1), ($\overline{2}$ +1,1), (1+1+1,1) (1+1+1,2) and (1+1+1,3), so that $\overline{spt}(3) = 6$.

3. The Generating Function for $\overline{spt_2}$ (*n*)

The generating function [5] for $\overline{spt}(n)$ is given by;

$$\sum_{n=1}^{\infty}\frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty}$$

$$=\frac{x\ (-x^2;x)_\infty}{(1-x)^2(x^2;x)_\infty}+\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty}+\ldots$$

$$=\frac{x(1+x^2)(1+x^3)\ldots}{(1-x)^2(1-x^2)(1-x^3)\ldots}+\frac{x^2(1+x^3)(1+x^4)\ldots}{(1-x^2)^2(1-x^3)(1-x^4)\ldots}+\ldots$$

$$=1.x+3.x^2+6.x^3+13.x^4+22.x^5+42.x^6+\ldots$$

$$=\overline{spt}(1)x+\overline{spt}(2)x^2+\overline{spt}(3)x^3+\overline{spt}(4)x^4+\ldots$$

$$=\sum_{n=1}^{\infty}\ \overline{spt}(n)x^n.$$

From above we get; $\overline{spt}(3)=6,\ \overline{spt}(6)=42,\ldots$ i.e., $\overline{spt}(3.1)=6\equiv 0 \pmod 3$, $\overline{spt}(3.2)=42\equiv 0 \pmod 3$, ...

We can conclude that;

$\overline{spt}(3n)\equiv 0 \pmod 3$, for $n \ge o$ [4].

Corollary 1: $\overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m,n).$

Proof: The generating function for $M_{\overline{S}}(m,n)$ is given by;

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M_{\overline{S}}(m,n)\ z^m x^n = \sum_{n=1}^{\infty}\frac{x^n(x^{n+1};x)_\infty(-x^{n+1};x)_\infty}{(zx^n;x)_\infty(z^{-1}x^n;x)_\infty}.$$

If z = 1, then we get;

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M_{\overline{S}}(m,n)\ x^n = \sum_{n=1}^{\infty}\frac{x^n(x^{n+1};x)_\infty(-x^{n+1};x)_\infty}{(x^n;x)_\infty(x^n;x)_\infty}$$

$$=\frac{x(x^2;x)_\infty(-x^2;x)_\infty}{(x;x)_\infty^2}+\frac{x^2(x^3;x)_\infty(-x^4;x)_\infty}{(x^2;x)^2_\infty}+\ldots$$

$$=\frac{x(-x^2;x)_\infty(1-x^2)(1-x^3)\ldots}{(1-x)^2(1-x^2)^2(1-x^3)^2\ldots}+\frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2(1-x^4)^2\ldots}+\ldots$$

$$=\frac{x(-x^2;x)_\infty}{(1-x)^2(1-x^2)(1-x^3)\ldots}+\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots}+\ldots$$

$$=\frac{x(-x^2;x)_\infty}{(1-x)^2(x^2;x)_\infty}+\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty}+\ldots$$

$$=\sum_{n=1}^{\infty}\frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty}=\sum_{n=1}^{\infty}\ \overline{spt}(n)x^n.$$

i.e., $\sum_{n=1}^{\infty} \overline{spt}(n)x^n = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m,n)\,x^n$.

Now equating the co-efficient of x^n from both sides we get;

$\overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m,n)$. Hence the Corollary.

Result 1: $M_{\overline{S}}(0,3,3) = M_{\overline{S}}(1,3,3) = M_{\overline{S}}(2,3,3) = \frac{1}{3}\overline{spt}(3)$.

Proof: We prove the result with an example. We see the vector partitions from $\overline{S}$ of 3 along with their weights and cranks are given as follows:

Table 1. *The vector partitions from $\overline{S}$ of 3 along with their weights and cranks.*

$\overline{S}$ -vector partition $(\vec{\pi})$ of 3	Weight $\omega(\vec{\pi})$	Crank $c(\vec{\pi})$
$\vec{\pi}_1 = (1,\phi,\phi,2)$	+ 1	0
$\vec{\pi}_2 = (1,\phi,1+1,\phi)$	+1	–2
$\vec{\pi}_3 = (1,1+1,\phi,\phi)$	+1	2
$\vec{\pi}_4 = (1,1,1,\phi)$	+1	0
$\vec{\pi}_5 = (1,\phi,2,\phi)$	+1	–1
$\vec{\pi}_6 = (1,2,\phi,\phi)$	+1	1
$\vec{\pi}_7 = (1+2,\phi,\phi,\phi)$	–1	0
$\vec{\pi}_8 = (3,\phi,\phi,\phi)$	+1	0
	$\sum\omega(\vec{\pi}) = 6$	

Here we have used ϕ to indicate the empty partition. Thus we have,

$M_{\overline{S}}(0,3,3) = +1+1-1+1 = 2$,
$M_{\overline{S}}(1,3,3) = M_{\overline{S}}(-2,3,3) = 1+1 = 2$,

$M_{\overline{S}}(2,3,3) = M_{\overline{S}}(-1,3,3) = 1+1 = 2$.

$\therefore M_{\overline{S}}(0,3,3) = M_{\overline{S}}(1,3,3) = M_{\overline{S}}(2,3,3) = 2 = \frac{1}{3}.6 = \frac{1}{3}\overline{spt}(3)$.
Hence the Result.

Now from above table we get;

$\sum\omega(\vec{\pi}) = 6$

$\sum_{k=0}^{2} M_{\overline{S}}(k,3,3) = 6$

$\therefore \overline{spt}(3) = \sum_{k=0}^{2} M_{\overline{S}}(k,3,3) = \sum\omega(\vec{\pi})$.

We define, $M_{\overline{S}}(k,t,m) = \sum_{m\equiv k(\mathrm{mod}\, t)} M_{\overline{S}}(m,n)$

and $\overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m,n) = \sum_{k=0}^{t-1} M_{\overline{s}}(k,t,n)$.

Theorem 1: The number of vector partitions of *n* in $\overline{S}$ with crank *m* counted according to the weight ω is non-negative, i.e., $M_{\overline{S}}(m,n) \geq 0$.

Proof: The generating function for $M_{\overline{S}}(m,n)$ is given by;

$$\sum_{n=1}^{\infty}\sum_{m} M_{\overline{S}}(m,n)z^m x^n$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{n+1};x)_\infty (-x^{n+1};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}.(x^{n+1};x)_\infty(-x^{n+1};x)_\infty$$

$$= \sum_{n=1}^{\infty} \frac{x^n \left(x^{2n+2};x^2\right)_\infty}{\left(zx^n;x\right)_\infty \left(z^{-1}x^n;x\right)_\infty}$$

[Since, $\sum_{n=1}^{\infty} (x^{n+1};x)_\infty(-x^{n+1};x)_\infty$

$$= (x^2;x)_\infty(-x^2;x)_\infty + (x^3;x)_\infty(-x^3;x)_\infty + (x^4;x)_\infty(-x^4;x)_\infty + \ldots$$

$$= (1-x^2)(1-x^3)\ldots(1+x^2)(1+x^3)\ldots + (1-x^3)(1-x^4)\ldots(1+x^3)(1+x^4)\ldots$$

$$= (1-x^4)(1-x^6)\ldots + (1-x^6)(1-x^8)+\ldots = \sum_{n=1}^{\infty}(x^{2n+2};x^2)_\infty]$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}.\frac{(x^{2n+2};x^2)}{(x^{2n};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}.\frac{1}{(1-x^{2n})(x^{2n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty} \frac{(x^{2n+2};x^2)_\infty}{(x^{2n};x)_\infty} = \frac{(x^4;x^2)_\infty}{(x^2;x)_\infty} + \frac{(x^6;x^2)_\infty}{(x^4;x)_\infty} + \ldots$

$$= \frac{(1-x^4)(1-x^6)\ldots}{(1-x^2)(1-x^3)(1-x^4)\ldots} + \frac{(1-x^6)(1-x^8)\ldots}{(1-x^4)(1-x^5)\ldots}$$

$$= \frac{1}{(1-x^2)}.\frac{1}{(1-x^3)(1-x^5)\ldots} + \frac{1}{(1-x^4)}.\frac{1}{(1-x^5)(1-x^7)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{1-x^{2n}}.\frac{1}{(x^{2n+1};x^2)_\infty}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}.\frac{1}{(1-x^{2n})(x^{2n+1};x^2)_\infty}$$

$$=\sum_{n=1}^{\infty}x^n\sum_{k=0}^{\infty}\frac{(z^{-1}x^n)^k}{(zx^{n+k};x)_\infty(x)_k}\cdot\frac{1}{(1-x^{2n})(x^{2n+1};x^2)_\infty}$$

[since $=\sum_{n=1}^{\infty}\dfrac{x^n.(x^{2n};x)_\infty}{(zx^n;x)_\infty(z^{-1}x^n;x)_\infty}=\sum_{n=1}^{\infty}x^n\sum_{k=0}^{\infty}\dfrac{(z^{-1}x^n)^k}{(zx^{n+k};x)_\infty(x)_k}$] (by [3]).

We see that the co-efficient of any power x in right hand side is non-negative so the co-efficient $M_{\overline{S}}(m,n)$ of $z^m x^n$ is non-negative, i.e., $M_{\overline{S}}(m,n)\geq 0$. Hence the Theorem.

Numerical Example 1: The vector partitions from $\overline{S}$ of 4 along with their weights and cranks are given as follows:

Table 2. *The vector partitions from $\overline{S}$ of 4 along with their weights and cranks.*

$\overline{S}$-vector partition $(\vec{\pi})$ of 4	Weight $\omega(\vec{\pi})$	Crank c$(\vec{\pi})$
$\vec{\pi}_1=(4,\phi,\phi,\phi)$	+ 1	0
$\vec{\pi}_2=(3+1,\phi,\phi,\phi)$	–1	0
$\vec{\pi}_3=(1,3,\phi,\phi)$	+1	1
$\vec{\pi}_4=(1,\phi,3,\phi)$	+1	–1
$\vec{\pi}_5=(1,\phi,\phi,3)$	+1	0
$\vec{\pi}_6=(2,2,\phi,\phi)$	+1	1
$\vec{\pi}_7=(2,\phi,2,\phi)$	+1	–1
$\vec{\pi}_8=(1+2,1,\phi,\phi)$	–1	1
$\vec{\pi}_9=(1+2,\phi,1,\phi)$	–1	–1
$\vec{\pi}_{10}=(1,1,2,\phi)$	+1	0
$\vec{\pi}_{11}=(1,2,1,\phi)$	+1	0
$\vec{\pi}_{12}=(1,1,\phi,2)$	+1	1
$\vec{\pi}_{13}=(1,\phi,1,2)$	+1	–1
$\vec{\pi}_{14}=(1,1+2,\phi,\phi)$	+1	2
$\vec{\pi}_{15}=(1,\phi,1+2,\phi)$	+1	–2
$\vec{\pi}_{16}=(1,1+1+1,\phi,\phi)$	+1	3
$\vec{\pi}_{17}=(1,\phi,1+1+1,\phi)$	+1	–3
$\vec{\pi}_{18}=(1,1+1,1,\phi)$	+1	1
$\vec{\pi}_{19}=(1,1,1+1,\phi)$	+1	–1
	$\sum\omega(\vec{\pi})=13$	

Here we have used ϕ to indicate the empty partition. Thus we have;

$M_{\overline{S}}(0,4)=3,\ M_{\overline{S}}(1,4)=3,\ M_{\overline{S}}(-1,4)=3,\ M_{\overline{S}}(2,4)=1,$
$M_{\overline{S}}(-2,4)=1,\ M_{\overline{S}}(3,4)=1,$ and $M_{\overline{S}}(-3,4)=1$.

$\therefore\sum_m M_{\overline{S}}(m,4)=13,$ i.e., every term is non-negative.

$\therefore$ $M_{\overline{S}}(m,4))\geq 0.$ But we have already found that $\sum_m M_{\overline{S}}(m,3)=6,$

i.e., every term is non-negative. $\therefore$ $M_{\overline{S}}(m,3)\geq 0.$

So we can conclude that; $M_{\overline{S}}(m,n)\geq 0.$

Corollary 2: $\overline{S}(1,x)=\sum_{n=1}^{\infty}\overline{spt}(n)x^n$.

Proof: We get $\overline{S}(z,x)=\sum_{n=1}^{\infty}\dfrac{x^n(-x^{n+1};x)_\infty(x^{n+1};x)_\infty}{(zx^n;x)_\infty(z^{-1}x^n;x)_\infty}$ ([2]).

If $z=1$, then we get;

$$\overline{S}(1,x)=\sum_{n=1}^{\infty}\frac{x^n(-x^{n+1};x)_\infty(x^{n+1};x)_\infty}{(x^n;x)_\infty(x^n;x)_\infty}$$

$$=\frac{x(-x^2;x)_\infty(x^2;x)_\infty}{(x;x)^2{}_\infty}+\frac{x^2(-x^3;x)_\infty(x^3;x)_\infty}{(x^2;x)^2{}_\infty}+\ldots$$

$$=\frac{x(-x^2;x)_\infty(1-x^2)(1-x^3)(1-x^4)\ldots}{(1-x)^2(1-x^2)^2(1-x^3)^2\ldots}+$$

$$\frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\left(1-x^4\right)^2\ldots}+\ldots$$

$$=\frac{x(-x^2;x)_\infty}{(1-x)^2(1-x^2)(1-x^3)\ldots}+\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots}+\ldots$$

$$=\sum_{n=1}^{\infty}\frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty}\overline{spt}\,(n)x^n,$$

i.e., $\overline{S}(1,x)=\sum_{n=1}^{\infty}\overline{spt}(n)x^n$. Hence the Corollary.

Theorem 2: $\overline{spt}\,(n)=\sum_{\substack{\vec{\lambda}\in\overline{SP}\\ |\vec{\lambda}|=|\lambda_1|+|\lambda_2|=n}}1$.

Proof: First we define the $\overline{sptcrank}$ in terms of partition pairs,

$\overline{SP}=\{\vec{\lambda}=(\lambda_1,\lambda_2)\in P\times P:1<s(\lambda_1)\leq s(\lambda_2)$ and all parts of λ_2 that are $\geq 2s(\lambda_1)+1$ are odd$\}$.

The generating function for $\overline{spt}(n)$ is given by;

$$\sum_{n=1}^{\infty}\overline{spt}(n)x^n=\sum_{n=1}^{\infty}\frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{(1-x^n)^2(x^{n+1};x)_\infty}(-x^{n+1};x)_\infty$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{(1-x^n)^2(x^{n+1};x)_\infty}\cdot\frac{(x^{2n+2};x^2)_\infty}{(x^{n+1};x)_\infty}$$

[Since, $\sum_{n=1}^{\infty}(-x^{n+1};x)_{\infty} = (-x^2;x)_{\infty} + (-x^3;x)_{\infty} + \ldots$

$$= (1+x^2)(1+x^3)\ldots + (1+x^3)(1+x^4)\ldots + (1+x^4)\ldots$$

$$= \frac{(1-x^4)(1-x^6)\ldots}{(1-x^2)(1-x^3)(1-x^4)\ldots} + \frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots} + \ldots$$

$$= \frac{(x^4;x^2)_{\infty}}{(x^2;x)_{\infty}} + \frac{(x^6;x^2)_{\infty}}{(x^3;x)_{\infty}} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(x^{2n+2};x^2)_{\infty}}{(x^{n+1};x)_{\infty}}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}(1-x^n)} \cdot \frac{(x^{2n+2};x^2)_{\infty}}{(x^{n+1};x)_{\infty}}$$

[Since, $\sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1};x)_{\infty}}$

$$= \frac{x}{(1-x)^2(x^2;x)_{\infty}} + \frac{x^2}{(1-x^2)^2(x^3;x)_{\infty}} + \ldots$$

$$= \frac{x}{(1-x)^2(1-x^2)(1-x^3)\ldots} + \frac{x^2}{(1-x^2)(1-x^3)(1-x^4)\ldots} + \ldots$$

$$= \frac{x}{(1-x)(x;x)_{\infty}} + \frac{x^2}{(1-x^2)(x^2;x)_{\infty}} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)(x^n;x)_{\infty}}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}} \cdot \frac{1}{(1-x^n)} \cdot \frac{(x^{2n+2};x^2)_{\infty}}{(x^{n+1};x)_{\infty}}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}} \cdot \frac{1}{(1-x^n)} \cdot \frac{1}{(1-x^{n+1})\ldots(1-x^{2n})(x^{2n+1};x^2)_{\infty}}$$

[Since, $\sum_{n=1}^{\infty} \frac{(x^{2n+2};x^2)_{\infty}}{(x^{n+1};x)_{\infty}} = \frac{(x^4;x^2)_{\infty}}{(x^2;x)_{\infty}} + \frac{(x^6;x^2)_{\infty}}{(x^3;x)_{\infty}} + \ldots$

$$= \frac{(1-x^4)(1-x^6)\ldots}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)\ldots} + \frac{(1-x^6)(1-x^4)\ldots}{(1-x^3)(1-x^4)(1-x^5)\ldots} + \ldots$$

$$= \frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)\ldots} + \frac{1}{(1-x^3)(1-x^4)(1-x^5)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(1-x^{n+1})\ldots(1-x^{2n})(x^{2n+1};x^2)_{\infty}}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}} \cdot \frac{1}{(1-x^n)(1-x^{n+1})\ldots(1-x^{2n})(x^{2n+1};x^2)_{\infty}}$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{\lambda_1 \in P \\ S(\lambda_1)=n}} x^{|\lambda_1|} \sum_{\substack{\lambda_2 \in P \\ s(\lambda_2)\ge n}} x^{|\lambda_2|}$$, all parts in $\lambda_2 \ge 2n+1$ are odd

$$= \sum_{n=1}^{\infty} \sum_{\substack{\vec{\lambda} \in \overline{SP} \\ |\vec{\lambda}| = |\lambda_1| + |\lambda_2| = n}} x^{|\vec{\lambda}|} .$$

Equating the co-efficient of x^n from both sides we get;
$\overline{spt}\,(n) = \sum_{\substack{\vec{\lambda} \in \overline{SP} \\ |\vec{\lambda}| = |\lambda_1| + |\lambda_2| = n}} 1$. Hence the Theorem.

Numerical Example 2: The overpartitions of 3 with smallest parts not overlined are; 3, 2+1, $\bar{2}+1$, 1+1+1. Consequently, the number of smallest parts in the overpartitions of 3 with smallest part not overlined is given by; $\dot{3}$, $2+\dot{1}$, $\bar{2}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}$ so that $\overline{spt}(3) = 6$, i.e., there are 6 $\overline{SP}$ -partition pairs of 3 like;
$(3,\phi)$, $(2+1,\phi)$, $(1+1+1,\phi)$, $(1+1,1)$, $(1,1+1)$ and $(1,2)$.

Result 2: $M_{\bar{S}}(0,3,3) = M_{\bar{S}}(1,3,3) = M_{\bar{S}}(2,3,3) = \frac{\overline{spt}(3)}{3}$.

Proof: First we define a $\overline{crank}$ of partitions pairs $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$.

For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ we define;

$k(\vec{\lambda})$ =# of pairs j in λ_2 such that $s(\lambda_1) \le j \le 2\ \ s(\lambda_1) - 1$, and also define;

$$\overline{crank}(\vec{\lambda}) = \begin{cases} (\#\text{of parts of } \lambda_1 \ge s(\lambda_1) + k) - k; & \text{if } k > 0 \\ (\#\text{of parts of } \lambda_1) - 1; & \text{if } k = 0 \end{cases}$$

where $k = k(\vec{\lambda})$.

Table 3. *$\overline{SP}$ -partition pairs of 3 with $\overline{crank}$.*

$\overline{SP}$ -partition pair $\vec{\lambda} = (\lambda_1, \lambda_2)$	k	$\overline{crank}$	(mod3)
$(3,\phi)$	0	0	0
$(2+1,\phi)$	0	1	1
$(1+1+1,\phi)$	0	2	2
$(1+1,1)$	1	−1	2
$(1,1+1)$	2	−2	1
$(1,2)$	0	0	0

From the table 3 we get;

$$M_{\bar{S}}(0,3,3) = M_{\bar{S}}(1,3,3) = M_{\bar{S}}(2,3,3) = 2 = \frac{1}{3}6 = \frac{1}{3}\overline{spt}(3) .$$

Hence the Result.

4. The $\overline{sptcrank}$ of a Marked Overpartition

We want to describe the $\overline{sptcrank}$ of a marked overpartition [6]. To define the $\overline{sptcrank}$ of a marked overpartition we first need to define a function $k\ (m,\ n)$ for positive integers m and n such that $m \geq n+1$ we write $m = b2^j$, where b is odd and $j \geq o$. For a given odd integer b and a positive integer n we define $j_0 = j_0(b,n)$ to be the smallest nonnegative integer j_0 such that $b2^{j_0} \geq n+1$.

Table 4. *Marked overpartitions of 6 with* $\overline{sptcrank}$ *(mod 3).*

Marked overpartition (π, j) of 6	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1)))$	$\bar{k}$	$\overline{sptcrank}$	**(mod3)**
(6,1)	6	ϕ	1	0	0	0	0
$(\bar{5}+1,1)$	1	5	1	0	0	0	0
$(\bar{4}+2,1)$	2	4	1	0	0	0	0
$(4+1+1,1)$	4+1+1	ϕ	2	0	1	0	0
$(\bar{3}+\bar{2}+1,1)$	1	3+2	1	0	0	0	0
$(3+1+1+1,2)$	3+1+1+1	ϕ	3	0	1	0	0
$(3+1+1+1,3)$	3+1+1+1	ϕ	3	0	0	3	0
$(2+2+1+1,2)$	2+2+1+1	ϕ	2	0	0	3	0
$(\bar{2}+2+1+1,2)$	2+1+1	2	2	0	1	0	0
$(2+1+1+1+1,1)$	2+1+1+1+1	ϕ	4	0	3	–3	0
$(2+1+1+1+1,3)$	2+1+1+1+1	ϕ	4	0	1	0	0
$(\bar{2}+1+1+1+1,1)$	1+1+1+1	2	4	0	3	–3	0
$(\bar{2}+1+1+1+1,4)$	1+1+1+1	2	4	0	0	–3	0
$(1+1+1+1+1+1,3)$	1+1+1+1+1+1	ϕ	6	0	3	–3	0
$(5+1,,1)$	5+1	ϕ	1	0	0	1	1
$(4+2,1)$	4+2	ϕ	1	0	0	1	1
$(\bar{4}+1+1,2)$	1+1	4	2	0	0	1	1
(3+3,2)	3+3	ϕ	2	0	0	1	1
$(\bar{3}+2+1,1)$	2+1	3	1	0	0	1	1
$(3+\bar{2}+1,1)$	3+1	2	1	0	0	1	1
$(\bar{3}+1+1+1,1)$	1+1+1	3	3	0	2	–2	1
(2+2+2,1)	2+2+2	ϕ	3	0	2	–2	1
(2+2+1+1,1)	2+2+1+1	ϕ	2	0	1	1	1
(2+1+1+1+1,2)	2+1+1+1+1	ϕ	4	0	2	–2	1
(2+1+1+1+1,4)	2+1+1+1+1	ϕ	4	0	0	4	1
$(\bar{2}+1+1+1+1,2)$	1+1+1+1	2	4	0	2	–2	1
(1+1+1+1+1+1,1)	1+1+1+1+1+1	ϕ	6	0	5	–5	1
(1+1+1+1+1+1,4)	1+1+1+1+1+1	ϕ	6	0	2	–2	1
(4+1+1,2)	4+1+1	ϕ	2	0	0	2	2
$(\bar{4}+1+1,1)$	1+1	4	2	0	1	–1	2
$(3+3,1)$	3+3	ϕ	2	0	1	–1	2
(3+2+1,1)	3+2+1	ϕ	1	0	0	2	2
(3+1+1+1,1)	3+1+1+1	ϕ	3	0	2	–1	2
$(\bar{3}+1+1+1,2)$	1+1+1	3	3	0	1	–1	2
$(\bar{3}+1+1+1,3)$	1+1+1	3	3	0	0	2	2
(2+2+2,2)	2+2+2	ϕ	3	0	1	–1	2
(2+2+2,3)	2+2+2	ϕ	3	0	0	2	2
$(\bar{2}+2+1+1,2)$	2+1+1	2	2	0	0	2	2
$(\bar{2}+1+1+1+1,3)$	1+1+1+1	2	4	0	1	–1	2
(1+1+1+1+1+1,2)	(1+1+1+1+1+1)	ϕ	6	0	4	–4	2
(1+1+1+1+1+1,5)	(1+1+1+1+1+1)	ϕ	6	0	1	–1	2
(1+1+1+1+1+1,6)	(1+1+1+1+1+1)	ϕ	6	0	0	5	2

We define; $k(m,n)=\begin{cases}0, \text{if } b\geq 2n\\ 2^{j-j_0} \text{ if } b2^{j_0}<2n\\ 0, \text{if } b2^{j_0}=2n.\end{cases}$

For a marked overpartitions (π, j) we let π_1 be the partition formed by the non-overlined parts of π, π_2 be the partition (into distinct parts) formed by the overlined parts of π so that $s(\pi_2)>s(\pi_1)$, we define $\bar{k}(\pi,i)=\nu(\pi_1)-j+k(\pi_2,s(\pi_1))$, where $\nu(\pi_1)$ is the number of smallest parts of π_1. Now we can define;

$$\overline{sptcrank}(\pi,j)=\begin{cases}(\#\text{of parts of } \pi_1\geq s(\pi_1)-\bar{k}), \text{ if } \bar{k}=\bar{k}(\pi,j)>0\\ (\#\text{of parts of } \pi_1)-1; \text{ if } \bar{k}=\bar{k}(\pi,j)=0.\end{cases}$$

Corollary 3[8]: The residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of $3n$ into 3 equal classes.

Proof: We prove the corollary with the help of an example when n = 2. There are 42 marked overpartitions of 6 so that $\overline{spt}(6)=42$.

We see that the residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of $3n$ into 3 equal classes. Hence the Corollary.

5. Conclusion

In this study we have found the number of smallest parts in the overpartitions of n with smallest part not overlined for n=1, 2, 3, 4. We have shown the relation $\overline{spt}\,(3n)\equiv 0$ (mod3) for $n=1,2,...$ and have shown the result 1 with the help of vector partitions from $\overline{S}$ of 3 along with their weights and cranks. We have verified the Theorem 1 when n = 4 and have verified the Theorem 2 when n = 3. We have established the Corollary 3 with 42 marked overpartitions of 6.

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