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Stability and Bifurcation Analysis for the Dynamical Model of a New Three-dimensional Chaotic System

Wen Wang, Liangqiang Zhou*

Department of Mathematics, Nan University of Aeronautics and Astronautics, Nanjing, PR China

Email address
1981977804ww@sina.com (Wen Wang), zlqrex@sina.com (Liangqiang Zhou)

*Corresponding author

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Abstract
Chaotic systems have important applications in secure communications. Therefore, it is of great theoretical and practical significance to study the dynamics of this class of system. This paper is devoted to investigating the dynamic behaviors of a three dimensional chaotic system. With both analytical and numerical methods, the nonlinear dynamic characteristics including stability and bifurcations of this new three dimensional chaotic system are investigated in this paper. It is presented that this system has symmetry and invariance. All the equilibriums of the system and their stability are studied in detail for different values of system parameters. It is presented that there may exist three equilibriums for this system. The stability conditions of these equilibriums are obtained with the Routh–Hurwitz criterion. Using the Hopf bifurcation theorem and the first Lyapunov coefficient, the system condition and type of Hopf bifurcation for this system is obtained analytically. It is demonstrated that there may exist subcritical Hopf bifurcations under certain system parameters for this system. Using the Runge-Kutta method, numerical simulations including phase portraits and time history curves are also given, which verify the analytical results. The results obtained here can provide some guidance for the analysis and design of secure communication systems.

Keywords
Three Dimensional Chaotic System, Stability, Hopf Bifurcation

1. Introduction

In 1963, the famous meteorologist Lorenz discovered the chaotic attractor of the three-dimensional autonomous system for the first time in numerical experiments, and proposed the Lorenz chaotic system [1]. Since then, people have had great interest in the field. In 1999, Chen discovered a chaotic system similar to the Lorenz chaotic system but with different topologies, called the Chen system [2]. In 2002, Lü and Chen discovered a new chaotic system, named the Lü system [3-4], which transitioned from the Lorenz chaotic system to the Chen system. In 2004, Liu et al. proposed a new three-dimensional autonomous chaotic system called Liu system with nonlinear squared terms [5-6]. In 2004, Tigan et al. removed a linear term on the Lorenz chaotic system and proposed a new chaotic system, namely T chaos system [7]. Wang et al. made a series of researches and analyses on the T chaotic system, for example, a simple dynamic analysis of the new three-dimensional chaotic system is carried out by using phase diagram, bifurcation and Lyapunov exponent, and adaptive synchronization control and circuit simulation are also implemented to realize its control method [8]. The periodic orbits of the T-chaotic system were studied by calculation and numerical simulation, and the existence of the chaotic system was further confirmed [9]. The phase diagram of the fractional-order T-chaotic system is drawn and analyzed using the predictive correction method. In addition, the T-chaotic system has been further researched and explored [10]. In 2005, Qi et al. discovered the Qi chaotic system, that is, each equation of the system contains a class of nonlinear terms, which leads to more complex dynamic characteristics [11]. In 2007, Wang proposed a new three-dimensional quadratic continuous autonomous chaotic system with only one system parameter [12].

A new three-dimensional autonomous chaotic system is proposed (this paper studies this new three-dimensional system): the system has three chaotic parameters and three nonlinear terms, and the three nonlinear term cross product terms are yz, xz and xy, the basic dynamics of the system are studied by theoretical derivation and the first Lyapunov coefficient method [13]. At the same time, in order to better apply the system in real life, computer software numerical simulation is carried out to further explain the system exists objectively. Studying the dynamic behavior of such systems can provide new ideas for key system research and practical engineering.

2. Model and Analysis of New Three-dimensional Chaotic System

2.1. Dynamic Model of New Three-dimensional Chaotic System

Consider a new three-dimensional chaotic system [13], the dynamic equation is as follows:

$$\begin{cases} \dot{x} = a(y-x) + yz \\ \dot{y} = cx - xz \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

Where x, y and z are state variables, and a, b and c are system parameters.

2.2. Dynamic Analysis of the System

2.2.1. Symmetry and Invariance

Since the system remains unchanged under the transformation $\phi:(x,y,z)\to(-x,-y,z)$, the system is symmetric about the z-axis, and all parameters have no deformation. If Φ is the solution of the system, then $\phi\Phi$ is also the solution of the system.

2.2.2. The Equilibrium Points of the System and Stability Analysis

Letting the value of the right end of the equation (1) equal to 0, we can get the equilibrium points of the system as follows:

$$O(0,0,0), A_+\left(\sqrt{\frac{bc(a+c)}{a}}, \sqrt{\frac{abc}{a+c}}, c\right), A_-\left(-\sqrt{\frac{bc(a+c)}{a}}, -\sqrt{\frac{abc}{a+c}}, c\right)$$

The Jacobian matrix evaluated at the point (x, y, z) is

$$A = \begin{pmatrix} -a & a+z & y \\ c-z & 0 & -x \\ y & x & -b \end{pmatrix} \tag{2}$$

the characteristic polynomial is as follows:

$$f(\lambda) = \lambda^3 + (a+b)\lambda^2 + (ab - ac + az - cz + x^2 - y^2 + z^2)\lambda - abc + abz + ax^2 + axy - bcz + bz^2 - xyc + 2xyz \tag{3}$$

The system's Jacobian matrix evaluated at point $O(0,0,0)$ is:

$$A_0 = \begin{pmatrix} -a & a & 0 \\ c & 0 & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{4}$$

Its characteristic polynomial is $(\lambda+b)(\lambda^2 + a\lambda - ac) = 0$. Then when $a^2 + 4ac > 0$, the eigenvalue as follows:

When $b > 0$, if $a > 0, c < 0$, it is easy to know that the above three characteristic values are negative, so the equilibrium point $O(0,0,0)$ is stable; if $a > 0, c > 0$, we can obtain $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 < 0$, and the equilibrium point is saddle point, then the point is unstable; if $a < 0, c > 0$, know $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 > 0$, the equilibrium point is unstable. When $b < 0$, it is easy to know that the equilibrium point is unstable.

Consider the stability of the system at points A_+, A_- below. Since the system has symmetry invariance, we only need to consider the stability at point A_+. The Jacobian matrix of the system at point A_+ is:

$$A_1 = \begin{pmatrix} -a & a+c & \sqrt{\frac{abc}{a+c}} \\ 0 & 0 & -\sqrt{\frac{bc(a+c)}{a}} \\ \sqrt{\frac{abc}{a+c}} & \sqrt{\frac{bc(a+c)}{a}} & -b \end{pmatrix} \tag{5}$$

the characteristic equation is $f(\lambda) = \lambda^3 + (a+b)\lambda^2 + (ab + \frac{bc(a+c)}{a} - \frac{abc}{a+c})\lambda + 2abc + 2bc^2$.

Theorem 1 (Routh–Hurwitz criterion [14]) Letting $f(\lambda) = |\lambda E - A| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, where $a_i > 0 (i = 0,1,\cdots,n), a_0 = 1$, then the necessary and sufficient conditions for $f(\lambda)$ to be a Hurwitz polynomial are:

$$\Delta_1 = a_1 > 0, \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \ldots$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & a_{n-1} & 0 \\ a_{2n-1} & a_{2n-2} & \cdots & \cdots & a_n \end{vmatrix} = \Delta_{n-1}a_n > 0 \tag{6}$$

So the equilibrium point A_+ is stable, if and only if the following conditions are satisfied:

$$\begin{cases} a+b > 0 \\ (a+b)(ab + \frac{bc(a+c)}{a} - \frac{abc}{a+c}) - (2abc + 2bc^2) > 0 \\ 2abc + 2bc^2 > 0 \end{cases} \tag{7}$$

3. Hopf Bifurcation Analysis

According to the Hopf bifurcation theory [15], when the system (1) contains a pair of complex eigenvalues that satisfy the following conditions, the system will generate Hopf bifurcation:

$$\begin{cases} \mathrm{Re}(\lambda)\big|_{b=b_0} = 0 \\ \mathrm{Im}(\lambda)\big|_{b=b_0} \neq 0 \\ \dfrac{d}{db}\mathrm{Re}(\lambda)\big|_{b=b_0} \neq 0 \end{cases} \tag{8}$$

Where b_0 is the critical value of b when the system generates bifurcation.

The system has no eigenvalues similar to $\lambda = \pm\omega i(\omega > 0)$ at the origin $O(0,0,0)$, so Hopf bifurcation does not occur at the origin.

Letting $\lambda_2 = \omega i, \lambda_3 = -\omega i$ substituting $\lambda_2 = \omega i$ into $f(\lambda) = 0$, we can obtain:

$$\begin{aligned} &-i\omega^3 - (a+b)\omega^2 + (ab - ac + az - cz + x^2 - y^2 + z^2)i\omega - \\ &abc + abz + ax^2 + axy - bcz + bz^2 - xyc + 2xyz = 0 \end{aligned} \tag{9}$$

Separating the real and imaginary parts of the above formula, we can obtain:

$$\begin{cases} \omega^2 = ab - ac + az - cz + x^2 - y^2 + z^2 \\ \omega^2 = \dfrac{-abc + abz + ax^2 + axy - bcz + bz^2 - xyc + 2xyz}{a+b} \end{cases} \tag{10}$$

The parameters satisfy the following conditions:

$$\begin{cases} a + b > 0 \\ ab - ac + az - cz + x^2 - y^2 + z^2 > 0 \\ -abc + abz + ax^2 + axy - bcz + bz^2 - xyc + 2xyz > 0 \\ ab - ac + az - cz + x^2 - y^2 + z^2 = \\ \quad \dfrac{-abc + abz + ax^2 + axy - bcz + bz^2 - xyc + 2xyz}{a+b} \end{cases} \tag{11}$$

In addition, we can obtain:

$$b_0 = \frac{a\ (-a^3 + a^2c + 2ac^2 + c^3)}{a^3 + a^2c + 2ac^2 + c^3} \tag{12}$$

the eigenvalues corresponding to the equilibrium points are:

$$\lambda_1 = -(a+b), \lambda_{2,3} = \pm i\sqrt{\frac{2bc(a+c)}{a+b}} \tag{13}$$

In summary, when $b > b_0$, the equilibrium point is stable. while $b < b_0$, the equilibrium point becomes unstable.

In the following, we use the first Lyapunov coefficient to further discuss the supercriticality or subercriticality of the Hopf bifurcation. Suppose C^n is a linear space defined in the complex field C with inner product, for any vector $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, where $x_i, y_i \in C(i = 1, 2, \cdots, n)$, $\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i} y_i$. Introduce norm $\|x\| = \sqrt{\langle x, x\rangle}$, so that C^n is a Hilbert space. Consider the following nonlinear system:

$$\dot{x} = Ax + F(x), x \in R^n \tag{14}$$

Where $F(x) = o(\|x\|^2)$ is a smooth function, and it can be expanded into:

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x\|^4) \tag{15}$$

in which $B(x,x)$ and $C(x,x,x)$ are bilinear and trilinear functions, respectively.

In equation (1), If the matrix A has a pair of pure imaginary eigenvalues $\lambda_{2,3} = \pm i\omega(w > 0)$, let $q \in C^n$ be a complex eigenvector corresponding to the eigenvalue λ_2, then we can obtain $Aq = iwq, A\overline{q} = -iw\overline{q}$. Meanwhile, we introduce the adjoint vector $p \in C^n$, which satisfies $A^T p = -iwp, A^T \overline{p} = iw\overline{p}$ and $< p, q >= 1$.

Introducing the transformation as follows:

$$\begin{cases} X = x - x_{eq} \\ Y = y - y_{eq} \\ Z = z - z_{eq} \end{cases} \tag{16}$$

then the equilibrium point (x_{eq}, y_{eq}, z_{eq}) transform to the origin $(0,0,0)$.

According to the first Lyapunov coefficient theorem at the equilibrium point of the system [16]:

$$l_1(0) = \frac{1}{2\omega}\mathrm{Re}\Big(\langle p, C(q,q,\overline{q})\rangle - 2\big\langle p, B(q, A^{-1}B(q,\overline{q}))\big\rangle + \big\langle p, B(\overline{q}, (2i\omega E - A)^{-1}B(q,q))\big\rangle\Big) \tag{17}$$

Choosing the system parameters $a = 1, c = 1$, then we can obtain that $b_0 = \dfrac{3}{5}$ and $A_+ = \left(\sqrt{\dfrac{6}{5}}, \sqrt{\dfrac{3}{10}}, 1\right)^T$, as well as the Jacobian matrix of the system (1):

$$A=\begin{pmatrix} -1 & 2 & \sqrt{\frac{3}{10}} \\ 0 & 0 & -\sqrt{\frac{6}{5}} \\ \sqrt{\frac{3}{10}} & \sqrt{\frac{6}{5}} & -\frac{3}{5} \end{pmatrix} \tag{18}$$

Calculating the corresponding vectors p,q of matrix J satisfy $Aq=i\omega q$, $A^T p=-i\omega p$, and $\langle p,q\rangle=1$, we can obtain:

$$q=\left(\frac{8\mathrm{i}+\sqrt{6}}{\sqrt{5}(2+\mathrm{i}\sqrt{6})},\frac{2}{\sqrt{5}}\mathrm{i},1\right)^T,\bar{q}=\left(\frac{-8\mathrm{i}+\sqrt{6}}{\sqrt{5}(2-\mathrm{i}\sqrt{6})},-\frac{2}{\sqrt{5}}\mathrm{i},1\right)^T \tag{19}$$

$$p=\left(\frac{\sqrt{30}(\sqrt{6}\mathrm{i}-2)}{4(13\sqrt{6}\mathrm{i}-1)},\frac{\sqrt{5}(\sqrt{6}\mathrm{i}-2)^2(\sqrt{6}-7\mathrm{i})}{10(13\sqrt{6}\mathrm{i}-1)},-\frac{5(\sqrt{6}\mathrm{i}-2)^2}{4(13\sqrt{6}\mathrm{i}-1)}\right)^T \tag{20}$$

where $\bar{q}$ is the conjugate vector of q.

In addition, the bilinear and trilinear functions for the system:

$$B(X,X')=(yz',-xz',xy')^T, C(X,X',X'')=(0,0,0)^T \tag{21}$$

then:

$$B(q,q)=\left(\frac{2}{\sqrt{5}}\mathrm{i},-\frac{8\mathrm{i}+\sqrt{6}}{\sqrt{5}(2+\mathrm{i}\sqrt{6})},\frac{2(-8+\mathrm{i}\sqrt{6})}{5(2+\mathrm{i}\sqrt{6})}\right)^T,\ B(q,\bar{q})=\left(\frac{2}{\sqrt{5}}\mathrm{i},-\frac{8\mathrm{i}+\sqrt{6}}{\sqrt{5}(2+\mathrm{i}\sqrt{6})},\frac{2(8-\mathrm{i}\sqrt{6})}{5(2+\mathrm{i}\sqrt{6})}\right)^T \tag{22}$$

The inverse of matrix A is:

$$A^{-1}=\begin{pmatrix} -\frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{30}}{6} \\ \frac{1}{4} & -\frac{1}{8} & \frac{\sqrt{30}}{12} \\ 0 & -\frac{\sqrt{30}}{6} & 0 \end{pmatrix} \tag{23}$$

Considering $F=A^{-1}B(q,\bar{q})$, one can obtain that:

$$F=\left(\frac{\sqrt{5}(24\mathrm{i}+53\sqrt{6})}{60(\mathrm{i}\sqrt{6}+2)},\frac{\sqrt{5}(24\mathrm{i}+23\sqrt{6})}{120(\mathrm{i}\sqrt{6}+2)},\frac{\sqrt{6}(8\mathrm{i}+\sqrt{6})}{6(\mathrm{i}\sqrt{6}+2)}\right)^T \tag{24}$$

and,

$$B(q,F)=\left(\frac{\sqrt{30}(\sqrt{6}\mathrm{i}-8)}{15(\sqrt{6}\mathrm{i}+2)},-\frac{(8\mathrm{i}+\sqrt{6})^2}{\sqrt{30}(\sqrt{6}\mathrm{i}+2)^2},\frac{(8\mathrm{i}+\sqrt{6})(24\mathrm{i}+23\sqrt{6})}{120(\sqrt{6}\mathrm{i}+2)^2}\right)^T \tag{25}$$

Therefore, it can be obtained that:

$$<p,B(q,F)>=-\frac{5(8\sqrt{6}\mathrm{i}+69)}{16(13\sqrt{6}\mathrm{i}+1)} \tag{26}$$

Due to $G=(2i\omega E-A)^{-1}B(q,q)$, it can be deduced that:

$$G=\begin{pmatrix} \frac{14\sqrt{5}(48\mathrm{i}-19\sqrt{6})}{45(5\sqrt{6}\mathrm{i}+8)(\sqrt{6}\mathrm{i}+2)} \\ -\frac{\sqrt{5}(253\sqrt{6}+504\mathrm{i})}{45(5\sqrt{6}\mathrm{i}+8)(\sqrt{6}\mathrm{i}+2)} \\ \frac{8(14\sqrt{6}\mathrm{i}-27)}{9(5\sqrt{6}\mathrm{i}+8)(\sqrt{6}\mathrm{i}+2)} \end{pmatrix} \tag{27}$$

and,

$$B(\bar{q},G)=\begin{pmatrix} -\frac{16\sqrt{5}\mathrm{i}(14\sqrt{6}\mathrm{i}-27)}{45(5\sqrt{6}i+8)(\sqrt{6}i+2)} \\ \frac{4\sqrt{5}(8\mathrm{i}-\sqrt{6})(14\sqrt{6}\mathrm{i}-27)}{225(5\sqrt{6}\mathrm{i}+8)} \\ \frac{(8\mathrm{i}-\sqrt{6})(253\sqrt{6}+504\mathrm{i})}{450(5\sqrt{6}\mathrm{i}+8)} \end{pmatrix} \tag{28}$$

Consequently, it is obtained that:

$$<p,B(\bar{q},G)>=-\frac{1175(2\sqrt{6}i+3)}{18(13\sqrt{6}i+1)(5\sqrt{6}i+8)} \tag{29}$$

$$l_1(0)=\frac{1}{2\omega}\mathrm{Re}\left(\langle p,C(q,q,\bar{q})\rangle-2\left\langle p,B(q,A^{-1}B(q,\bar{q}))\right\rangle+\left\langle p,B(\bar{q},(2i\omega E-A)^{-1}B(q,q))\right\rangle\right)=\frac{1}{\sqrt{6}}\cdot\frac{65691}{173768}\approx 0.154334\ >0 \tag{30}$$

Therefore, Hopf bifurcation obtained under this set of parameters is subcritical.

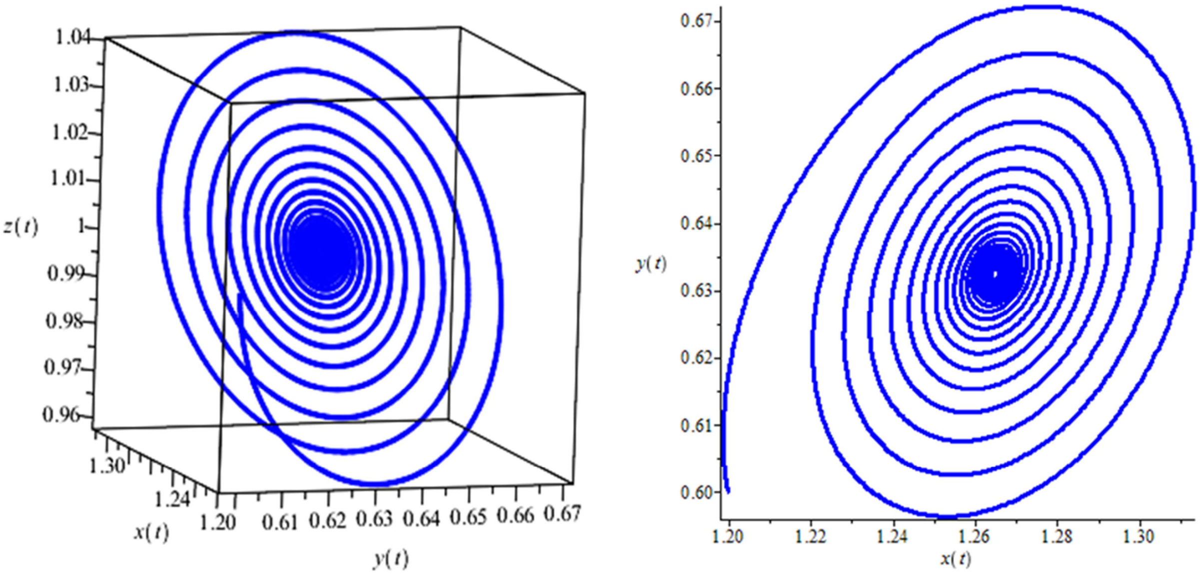
4. Numerical Simulations

We numerically simulate the system (1) by using Runge-Kutta numerical calculation method. For the system, choosing the following parameters:

$$a=1, c=1, x=1.2, y=0.6, z=1$$

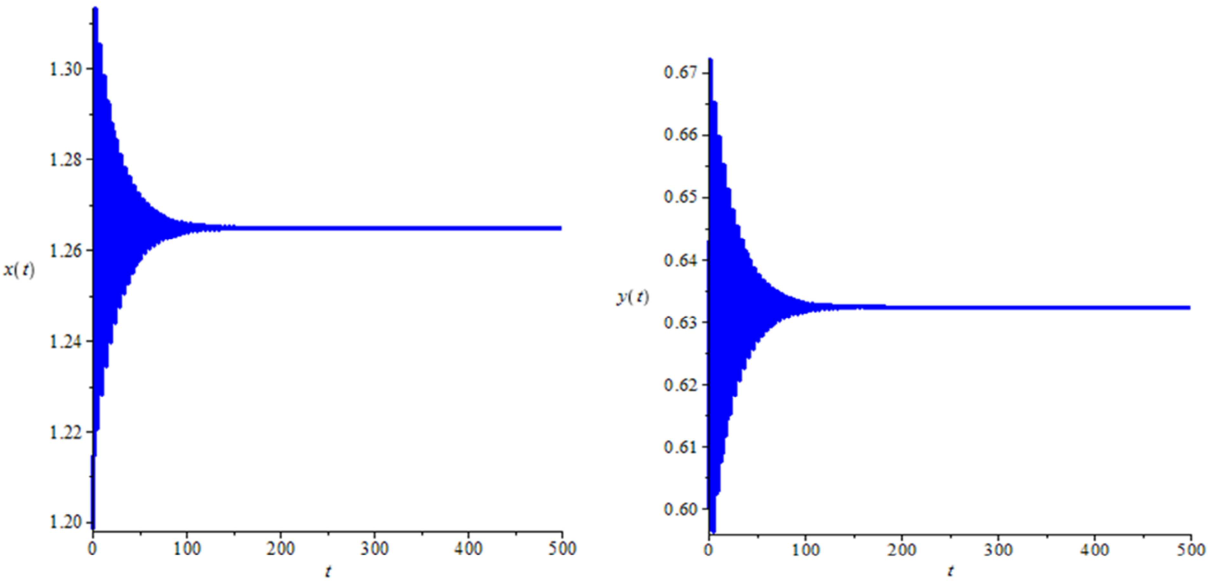
thus, when the branching occurs, the critical value is $b_0=\frac{3}{5}$.

1. When $b=0.8$, then $b>b_0$, the phase portrait of $x(t)-y(t)-z(t)$ and $x(t)-y(t)$ are shown as in Figure 1 (a), (b), respectively. In addition, Trajectories of $x(t)$ are $y(t)$ shown as in Figure 2 (a), (b), respectively, form which we can see that the equilibrium is stable.



(a) the phase portrait of x(t) y(t) z(t) (b) the phase portrait of x(t) y(t)

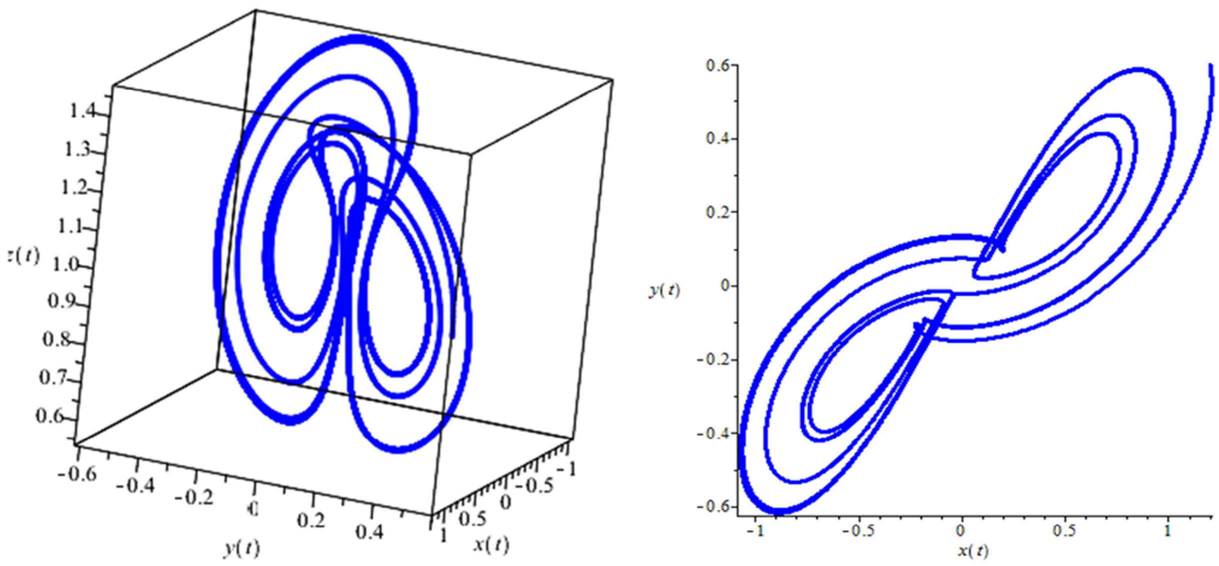
Figure 1. *The phase portrait for* $a=1, c=1, b=0.8$.



(a) The trajectories of x(t) (b) The trajectories of y(t)

Figure 2. *Trajectories for* $a=1, c=1, b=0.8$.

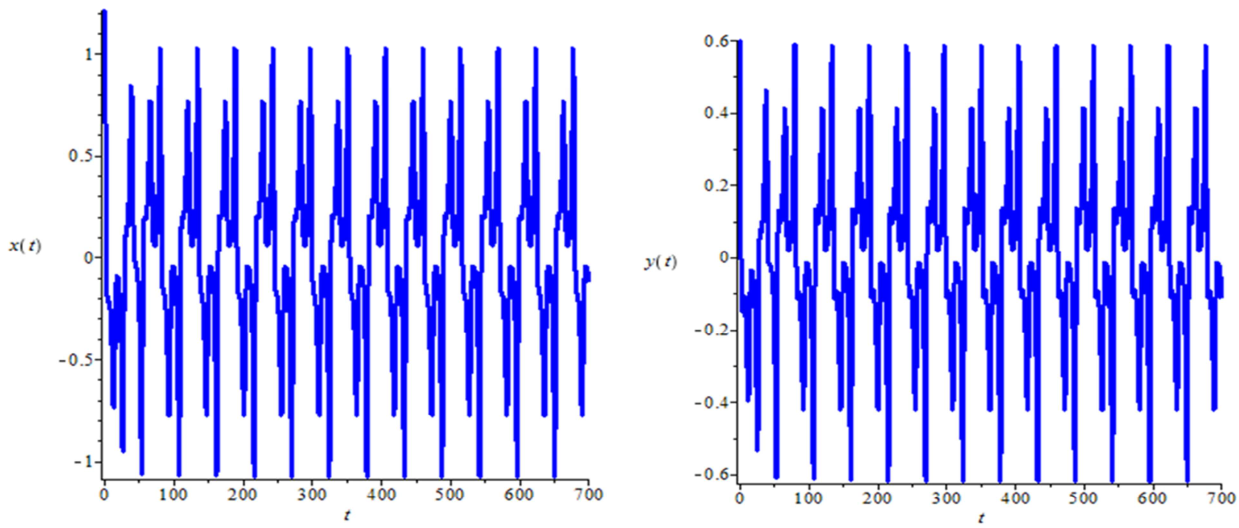
2. When $b=0.1$, then $b<b_0$, we can get the graph shown as in Figure 3 and Figure 4, respectively.



(a) The phase portrait of x(t) y(t) z(t)

(b) The phase portrait of x(t) y(t)

Figure 3. *The phase portrait for* $a=1, c=1, b=0.1$.



(a) the trajectories of x(t)

(b) the trajectories of y(t)

Figure 4. *Trajectories for* $a=1, c=1, b=0.1$.

It can be seen from the previous analysis that when $b > b_0$, the equilibrium point is asymptotically stable, and when $b < b_0$, the system will generate Hopf bifurcation, and the numerical results are consistent with the theoretical analysis results.

5. Conclusion

Nonlinear dynamics of a new three-dimensional chaotic system is investigated in this paper. The stability of the equilibrium point is discussed. Hopf bifurcation theorem and the first Lyapunov coefficient theorem are used to investigate the conditions and types of bifurcation in the system. It is presented that there may exisit subcritical Hopf bifurcations under certain system parameters for this system. Numerical simulations verify the analytical results. The results can provide some inspiration and guidance for the design of system parameters in secure communication systems. For instance, we can choose the system parameters so that there exist complex dynamic behaviors, such as bifurcations and chaos, which can enhance the confidentiality of communication, etc.

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