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On (*h*, *k*)-decay of evolution operators in Banach spaces

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Abstract

The main aim of this work is to define and exemplify various decay concepts and to emphasize connections between them. These decay concepts are included in a general concept,the so-called (*h*, *k*)-decay. Some illustrating examples clarify the relations between these properties.

Keywords

Evolution Operator, (*h*, *k*)-Decay, Exponential Decay, Polynomial Decay

1. Introduction

In the mathematical literature of the last decades, the asymptotic properties of solutions of evolution equations in finite or infinite dimensional space have proved to be research area of large intensity. There were defined and developed concepts of the asymptotic behaviors, as stability, expansivity, dichotomy (see [1--16] and the references therein), based on the fact that the dynamical systems which describe processes from economics, physical sciences, or engineering are extremely complex and the identification of the proper mathematical model is difficult.

In this paper, let X be a real or complex Banach space. The norm on X and on $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X, will be denoted by $\|\cdot\|$. Let I be the identity operator on X and T be the set defined by $T=\left\{(t,s)\in\mathbb{R}_+^2 : t\ge s\ge 0\right\}$.

Definition 1.1.

An operator-valued function $U:T\to\mathcal{B}(X)$ is said to be an evolution operator on X iff the following relations hold:

(e1) $U(t,t)=I$ for every $t\ge 0$;

(e2) $U(t,s)U(s,t_0)=U(t,t_0)$ for all (t,s) and $(s,t_0)\in T$.

If $h,k:\mathbb{R}_+=[0,\infty)\to[1,\infty)$ then we introduce the concept of (h,k)-decay as follows:

Definition 1.2.

The evolution operator $U:T\to\mathcal{B}(X)$ is said to be with (h,k)-decay (and we denote this by (h,k)-d) iff there are $N\ge 1$ and $t_0>0$ such that

$$Nk(s)\left\|U(t,s)x\right\|\ge\frac{h(s)}{h(t)}\|x\| \tag{1}$$

for all $(t,s,x)\in T\times X$ with $s\ge t_0$.

In the next section some particular cases of (h,k)-decay are considered. Some illustrating examples clarify the connections between these stability concepts.

2. Exponential Decay

Let $U:T\to\mathcal{B}(X)$ be an evolution operator on

Banach space X.

Definition 2.1.

The evolution operator $U:T\to\mathcal{B}(X)$ is said to be with:

(i) uniform exponential decay (and denoted as u.e.d.) iff there are $N\ge 1$, $\alpha>0$ and $t_0>0$ such that

$$N\|U(t,s)x\|\ge e^{-\alpha(t-s)}\|x\| \tag{2}$$

for all $(t,s,x)\in T\times X$ with $s\ge t_0$;

(ii) exponential decay in the sense Barreira-Valls (and denoted as B.V.e.d.) iff there are $N\ge 1$, $\alpha>0$, $\beta\ge 0$ and $t_0>0$ such that

$$Ne^{\beta s}\|U(t,s)x\|\ge e^{-\alpha(t-s)}\|x\| \tag{3}$$

for all $(t,s,x)\in T\times X$ with $s\ge t_0$;

(iii) (nonuniform) exponential decay (and denoted as e.d.) iff there are $N\ge 1$, $\alpha>0$, $t_0>0$ and a function $k:\mathbb{R}_+\to[1,\infty)$ such that

$$Nk(s)\|U(t,s)x\|\ge e^{-\alpha(t-s)}\|x\| \tag{4}$$

for all $(t,s,x)\in T\times X$ with $s\ge t_0$.

Remark 2.2.

It is obvious that

u.e.d. $\Rightarrow$ B.V.e.d. $\Rightarrow$ e.d.,

but the converse implications are not necessarily valid. To show this, we consider the following two examples.

Example 2.3.

(Evolution operator with B.V.e.d and without u.e.d) Consider the evolution operator (on $X=\mathbb{R}$) $U:T\to\mathcal{B}(X), U(t,s)x=e^{r(t)-r(s)}x$, where

$r(t)=t\cos t-2t$.

Successively, we have

$$r(t)-r(s)\ge-(t-s)-2t\sin^2\frac{t}{2}+2s\sin^2\frac{s}{2}$$
$$\ge-(t-s)-2t, \forall t\ge s\ge 0.$$

Thus

$e^{2t}|U(t,s)x|\ge e^{-(t-s)}|x|, \forall(t,s,x)\in T\times\mathbb{R}$,

which is equivalent with

$e^{2s}|U(t,s)x|\ge e^{-3(t-s)}|x|, \forall(t,s,x)\in T\times\mathbb{R}$,

This shows that U has B.V.e.d.

On the other hand, if we assume a contradiction that U has uniform exponential decay. Letting $t=(2n+1)\pi, s=2n\pi, n\in\mathbb{N}$ and $x\in X$ with $\|x\|=1$ in Definition 2.1-(i), we deduce that

$Ne^{(\alpha-3)\pi}\ge e^{4n\pi}, \forall n\in\mathbb{N}$,

which is not true. Therefore, U has not u.e.d.

Example 2.4.

(Evolution operator with e.d and without B.V.e.d) Let $u:\mathbb{R}_+\to[1,\infty)$ be a continuous function with $u(n+1/n)=1$ and $u(n)=e^{n^2}$ for all $n\in\mathbb{N}^*$.

Then the evolution operator (on $\mathbb{R}$) defined by $U(t,s)=\dfrac{u(t)}{u(s)}e^{-(t-s)}$ satisfies the relation

$$k(s)|U(t,s)x|=[1+u(s)]|U(t,s)x|\ge e^{-(t-s)}|x|,$$
$$\forall t\ge s\ge 0, \forall x\in\mathbb{R}$$

and hence U has e.d.

If we suppose that U has B.V.e.d. then for $t=n+1/n$, $s=n, n\in\mathbb{N}^*$ and $x\in X$ with $\|x\|=1$ in Definition 2.1-(ii), we obtain a contradiction and hence U has not B.V.e.d.

Let $\mathcal{E}$ be the set of all functions $f:\mathbb{R}_+\to[1,\infty)$ with the property that there is an $\alpha\ge 0$ such that $f(t)=e^{\alpha t}$ for every $t\ge 0$.

Remark 2.5.

(i) U is with u.e.d. iff there are $h\in\mathcal{E}$ and $k=const.(\ge 1)$ such that U has (h,k)-decay;

(ii) U is with B.V.e.d. iff there are $h,k\in\mathcal{E}$ such that U has (h,k)-decay;

(iii) U is with e.d. iff there are $h\in\mathcal{E}$ and a function $k:\mathbb{R}_+\to[1,\infty)$ such that U has (h,k)-decay.

3. Polynomial Decay

Let $U:T\to\mathcal{B}(X)$ be an evolution operator on Banach space X.

Definition 3.1.

The evolution operator $U:T\to\mathcal{B}(X)$ is said to be with:

(i) uniform polynomial decay (and denoted as u.p.d.) iff there are $N\ge 1$, $\alpha>0$ and $t_0\ge 1$ such that

$$N\|U(t,s)x\|\ge t^{-\alpha}s^{\alpha}\|x\| \tag{5}$$

for all $(t,s,x)\in T\times X$ with $s\ge t_0\ge 1$;

(ii) polynomial decay in the sense Barreira-Valls (and denoted as B.V.p.d.) iff there are $N \geq 1$, $\alpha > 0$, $\beta \geq 0$ and $t_0 \geq 1$ such that

$$Ns^{\beta} \|U(t,s)x\| \geq t^{-\alpha} s^{\alpha} \|x\| \quad (6)$$

for all $(t,s,x) \in T \times X$ with $s \geq t_0 \geq 1$;

(iii) (nonuniform) polynomial decay (and denoted as p.d.) iff there are $N \geq 1$, $\alpha > 0$, $t_0 \geq 1$ and a function $k : \mathbb{R}_+ \to [1,\infty)$ such that

$$Nk(s) \|U(t,s)x\| \geq t^{-\alpha} s^{\alpha} \|x\| \quad (7)$$

for all $(t,s,x) \in T \times X$ with $s \geq t_0 \geq 1$.

Remark 3.2.

It is obvious that
u.p.d. $\Rightarrow$ B.V.p.d. $\Rightarrow$ p.d.
The following two examples show that the converse implications between this decay concepts are not valid.

Example 3.3.

(Evolution operator with B.V.p.d and without u.p.d) Let $u : \mathbb{R}_+ \to (0,\infty)$ be the function defined by $u(t) = (t+1)^{3-\sin\ln(t+1)}$.

Then $U : T \to \mathcal{B}(\mathbb{R}), U(t,s)x = \dfrac{u(t)}{u(s)} x$ is an evolution operator on $X = \mathbb{R}$ with

$$|U(t,s)x| \geq \frac{(t+1)^2}{(s+1)^4} \geq \frac{2t}{(2s)^4} \geq \frac{st^{-1}}{8s^5}$$

for all $(t,s,x) \in T \times X$ with $s \geq t_0 = 1$ and hence U is with B.V.p.d.

If we suppose that U has u.p.d. then there exist $N \geq 1$, $\alpha > 0$ and $t_0 \geq 1$ such that

$$N(t+1)^3 (s+1)^{\sin\ln(s+1)} \geq s^{\alpha} t^{-\alpha} (s+1)^3 (t+1)^{\sin\ln(t+1)}$$

for all $t \geq s \geq t_0$.

Then for $t = \exp(2n\pi + \pi/2) - 1$ and $s = \exp(2n\pi) - 1$ we obtain a contradiction.

Example 3.4.

(Evolution operator with p.d and without B.V.p.d) Let $u : \mathbb{R}_+ \to [1,\infty)$ be a continuous function with $u(n + 1/n) = e$ and $u(n) = e^n$ for all $n \in \mathbb{N}^*$.

Then the evolution operator (on $\mathbb{R}$) defined by $U(t,s)x = \dfrac{u(t)s^2}{u(s)t^2} x$ satisfies the relation

$$k(s)|U(t,s)x| = [1+u(s)]|U(t,s)x| \geq s^2 t^{-2} |x|$$

for all $(t,s,x) \in T \times X$ with $s \geq t_0 = 1$ and hence U is with p.d.

If we suppose that U has B.V.p.d. then for $t = n + 1/n, s = n$ in Definition 3.1-(ii), thus we obtain that

$$Nn^{\beta+2} e \geq n^{\alpha} (1 + 1/n)^{2-\alpha} e^n$$

for all $n \in \mathbb{N}^*$, which is false.

Let $\mathcal{P}$ be the set of all functions $f : \mathbb{R}_+ \to [1,\infty)$ with the property that there is an $\alpha \geq 0$ such that $f(t) = t^{\alpha}$ for every $t \geq 0$.

Remark 3.5.

(i) U is with u.p.d. iff there are $h \in \mathcal{P}$ and $k = const.(\geq 1)$ such that U has (h,k)-decay;

(ii) U is with B.V. p.d. iff there are $h, k \in \mathcal{P}$ such that U has (h,k)-decay;

(iii) U is with p.d. iff there are $h \in \mathcal{P}$ and a function $k : \mathbb{R}_+ \to [1,\infty)$ such that U has (h,k)-decay.

Proposition 3.6.

If the evolution operator U has u.p.d. then it is with u.e.d.

Proof.

It is immediate from Definition 2.1 and Definition 3.1, using the fact that the function $\varphi(t) = e^t / t$ is nondecreasing on $[1,+\infty)$. □

Proposition 3.7.

If the evolution operator U has p.d. then it is with e.d.

Example 3.8.

(Evolution operator with e.d and without p.d) We consider the function u and the evolution operator U on X defined as in Example 2.4.

It is obvious that U has e.d. If we suppose that U is p.d. then there exist $N \geq 1$
, $\alpha > 0$, and a function $k : \mathbb{R}_+ \to [1,\infty)$ such that

$$Nk(s)u(t)e^s \geq s^{\alpha} t^{-\alpha} u(s) e^t$$

for all $t \geq s \geq t_0 = 1$.

Then for $s = 2$ and $t = n + 1/n \, (n \geq 2)$ we obtain a contradiction.

Remark 3.9.

An evolution operator which is with u.e.d. (and hence B.V.e.d. and e.d.) which is not p.d. (and hence neither B.V.p.d. nor u.p.d.). This fact is illustrated by the following example.

Example 3.10.

We consider on $X = \mathbb{R}$ the evolution operator $U(t,s)x = e^{-(t-s)}x$.

Clearly, U is with u.e.d. If we suppose that U is p.d. then there exist $N \geq 1$, $\alpha > 0$, $t_0 \geq 1$ and a function $k : \mathbb{R}_+ \to [1,\infty)$ such that

$$Nk(s)e^s t^\alpha \geq s^\alpha e^t$$

for all $t \geq s \geq t_0$.

Then for $s = t_0 + 1$ and $t \to \infty$ we obtain a contradiction.

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