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On orthonormal Bernstein polynomial of order eight

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Abstract

In this paper, we present the new orthonormal base $OB_{i,8},\ i = 0,1,\dots,8$ through the Gram-Schmidt Orthonormalization process on Bernstein polynomials of order eight. Bernstein polynomials and their properties are employed to derive explicit formulas for derivative and integration operational matrices of orthonormal Bernstein polynomials of order eight. Convergence criteria is also included in this paper. The relationship between the derivative of $OB_{i,8}$ and $B_{i,8}$ themselves with some other important properties are derived in this work. All the proposed results are of direct interest in many applications.

Keywords

Bernstein Polynomials, Gram- Schmidt Orthonormalization Process, Operational Matrix of Derivative and Integration

1. Introduction

Bernstein polynomials have many important properties and have been used for solving different problems using some approximate methods. Bernstein polynomials have been used for solving Fredholm Integral equations [1], boundary value problem [2], differential equation [3], nonlinear differential equations with collocation method [4], the Emden-Fowler equations which is arising in astrophysics [5] and others see [6-10].

The Bernstein polynomials are not orthonormal so their uses in the least square approximation are limited. To overcome this difficulty, Gram- Schmidt orthonormalization process can be used to construct the orthonormal Bernstein polynomials.

In this paper, we first construct an orthonormal family $\{OB_{i,8}\}_{i=0}^{8}$ of polynomials of degree eight. Then orthonormal Bernstein operational matrices of both derivative and integration are derived. In addition, the convergence criteria of $\{OB_{i,8}\}_{i=0}^{8}$ with some important results between $OB_{i,8}$ and $B_{i,8}$ are presented.

2. Bernstein Polynomial and Their Properties

The Bernstein basis polynomials of degree n are defined by [1]

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \tag{1}$$

By using binomial expansion of $(1-x)^{n-i}$, we can get

$$B_{i,n}(x) = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} x^{i+k} \tag{2}$$

Then $\{B_{i,n}\}_{i=0}^{n}$ in Hilbert space $L^2[0,1]$ is a complete basis. That is any polynomial of degree n can be expanded in terms of linear combination of $\{B_{0,n}, B_{1,n}, \dots, B_{n,n}\}$. Suppose that the function $f \in C^{m+1}[0,1]$, and

$S_m = span\,\{B_{0n}, B_{1n}, \dots, B_{nn}\}$. If $C^T B$ be the best approximation f out of S then [11]

$\|f - C^T B\|_{L^2[0,1]} \le \frac{\hat{k}}{(m+1)!\,\sqrt{2m+3}}$, where $\hat{k} = \max_{\propto \in [0,1]} |f^{(m+1)}(x)|$.

3. Orthonormal Bernstein Polynomials of Order Eight

Given a set of linear independent functions $B_{i,8}, i = 0,1,\dots,8$, if we define the inner product

$$< f, g >= \int_0^1 f(x)g(x)dx\,.$$

and the norm

$$||f||^2 =< f, f >$$

One can derive the orthonormal base $\{OB_8\}$ through the Gram-Schmidt orthonormalization process as the following

$$\Psi_0(x) = B_0\ ,\ \ OB_0 = \Psi_0/||\Psi_0||,$$

$$\Psi_1(x) = B_1 - a_{10}OB_0$$

where we require

$$< \Psi_1, OB_0 > = < B_1, OB_0 > + a_{10} < OB_0, OB_0 > = 0$$

Which yields, $\quad < OB_0, OB_0 > = ||OB_0||^2 = 1$

$$a_{10} =< B_1, OB_0 >$$

Hence,

$$\Psi_1 = B_1 - < B_1, OB_0 > OB_0\ , OB_1 = \Psi_1/||\Psi_1||$$

By induction, one has

$$\Psi_8 = B_8 - \sum_{j=0}^{7} a_{8j}\, OB_j \qquad , OB_8 = \Psi_8/||\Psi_8||$$

where

$$a_{8j} =< B_8, OB_j > .$$

The normalized Bernstein polynomials are

$$OB_0 = \sqrt{17}(t-1)^8$$

$$OB_1 = \sqrt{60}\,[-8t(t-1)^7 - \frac{1}{2}(t-1)^8]$$

$$OB_2 = \frac{15\sqrt{13}}{4}[8t(t-1)^7 + \frac{4}{15}(t-1)^8 + 28t^2(t-1)^6]$$

$$OB_3 = \frac{13\sqrt{11}}{2}[\frac{-84}{13}t(t-1)^7 - \frac{2}{13}(t-1)^8 - 42t^2(t-1)^6 - 56t^3(t-1)^5]$$

$$OB_4 = \frac{143*3}{14}[\frac{56}{11}t(t-1)^7 + \frac{14}{143}(t-1)^8 + \frac{504}{11}t^2(t-1)^6 + 112t^3(t-1)^5 + 70t^4(t-1)^4]$$

$$OB_5 = \frac{99}{\sqrt{7}}[\frac{-140}{33}t(t-1)^7 - \frac{7}{99}(t-1)^8 - \frac{140}{3}t^2(t-1)^6 - 1400t^3(t-1)^5 - 175t^4(t-1)^4 - 56t^5(t-1)^3]$$

$$OB_6 = \frac{33\sqrt{5}}{2}[4t(t-1)^7 + \frac{2}{33}(t-1)^8 + 50t^2(t-1)^6 + 200t^3(t-1)^5 + 300t^4(t-1)^4 + 168t^5(t-1)^3 + 28t^6(t-1)^2\ ]$$

$$OB_7 = 15\sqrt{3}\left[-\frac{14}{3}t(t-1)^7 - 8t^7(t-1) - \frac{1}{15}(t-1)^8 - 63t^2(t-1)^6 - 280t^3(t-1)^5 - 490t^4(t-1)^4 - \frac{1764}{5}t^5(t-1)^3 - 98t^6(t-1)^2\right]$$

$$OB_8 = 9\left[8t(t-1)^7 + 32t^7(t-1) + \frac{1}{9}(t-1)^8 + 112t^2(t-1)^6 + \frac{1568}{3}t^3(t-1)^5 + 980t^4(t-1)^4 + 784t^5(t-1)^3 + \frac{784}{3}t^6(t-1)^2 + t^8\right]$$

The coefficients of $t^{n-i}(t-1)^i$ for the normalized Bernstein polynomials of order eight are listed in table (1).

Table (1). *Coefficients of* $t^{n-i}(t-1)^i$ *for the normalized Bernstein polynomials of order eight.*

	t^8	$t^7(t-1)$	$t^6(t-1)^2$	$t^5(t-1)^3$	$t^4(t-1)^4$	$t^3(t-1)^5$	$t^2(t-1)^6$	$t(t-1)^7$	$(t-1)^8$
OB_8	9	288	2352	7056	8820	4704	1008	72	1
OB_7	-	- 207.85	- 2.55e+3	- 9.17e+3	- 1.27e+4	- 7.27e+3	- 1.64e+3	- 121.24	- 1.7321
OB_6	-	-	1.03e+3	6.19e+3	1.11e+4	7.38e+3	1.85e+3	147.58	2.2361
OB_5	-	-	-	-2.095e+3	- 6.55e+3	- 5.82e+3	- 1.75e+3	- 158.75	- 2.6458
OB_4	-	-	-	-	2145	3432	1404	156	3
OB_3	-	-	-	-	-	-1.21e+3	-905.44	- 139.298	-3.3166
OB_2	-	-	-	-	-	-	378.583	108.167	3.6056
OB_1	-	-	-	-	-	-	-	-61.9677	-3.873
OB_0	-	-	-	-	-	-	-	-	4.1231

4. Convergence Criteria for OB_i

If the function $f(x)$ is expanded in terms of orthonormal Bernstein polynomials

$$f(x) = \textstyle\sum_{i=0}^{\infty} f_i OB_i(x) \qquad (3)$$

It is not possible to perform computation an infinite number of terms, therefore, the series in (3) must be truncated. In place of (3) we take:

$$f(x) = \textstyle\sum_{i=0}^{n} f_i OB_i(x) \qquad (4)$$

So that $f(x) = f_n(x) + \sum_{i=n+1}^{\infty} f_n\, OB_i(x)$

$$\text{or } f(x) - f_n(x) = r(x) \qquad (5)$$

where

$$r(x) = \sum_{i=n+1}^{\infty} f_n OB_i(x) \tag{6}$$

The coefficients in (4) and (5) must be selected such that the norm of the residual function $||r(x)||$ is less than some convergence criteria $\in$, that is $\left||r(x)|\right| <\in$

$$||r(x)||^2 = \int_0^1 [\sum_{i=0}^{n+m} f_i OB_i(x) - \sum_{i=0}^{n} f_i OB_i(x)]^2\, dx$$

$$= \int_0^1 [\sum_{i=n+1}^{n+m} f_i OB_i(x)]^2 dx$$

$$= \int_0^1 [\sum_{i=n+1}^{n+m} f_i OB_i(x)][\sum_{i=n+1}^{n+m} f_i OB_i(x)] dx$$

$$= \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} f_i f_j \int_0^1 OB_i(x) OB_j(x) dx$$

We have

$$\int_0^1 OB_i(x) OB_j(x) dx = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

Then

$$||r(x)||^2 = \sum_{i=n+1}^{n+m} f_i^2$$

That is

$$\sum_{i=n+1}^{n+m} f_i^2 <\in$$

5. Expansion of OB_8 in terms of B_8

There is a relation between OB_8 and B_8 in the form

$OB_0 = \sqrt{17} B_0$

$OB_1 = 2\sqrt{15}\left(B_1 - \frac{1}{2} B_0\right)$

$OB_2 = \frac{15\sqrt{13}}{4}\left(B_2 - B_1 + \frac{4}{15} B_0\right)$

$OB_3 = \frac{13\sqrt{11}}{2}\left(B_3 - \frac{3}{2} B_2 + \frac{21}{26} B_1 - \frac{2}{13} B_0\right)$

$OB_4 = \frac{143 * 3}{14}\left(B_4 - 2B_3 + \frac{18}{11} B_2 - \frac{7}{11} B_1 + \frac{14}{143} B_0\right)$

$OB_5 = \frac{99}{\sqrt{7}}\left(B_5 - \frac{5}{2} B_4 + \frac{25}{9} B_3 - \frac{5}{4} B_2 + \frac{35}{66} B_1 - \frac{7}{99} B_0\right)$

$OB_6 = \frac{33\sqrt{5}}{2}\left(B_6 - 3B_5 + \frac{30}{7} B_4 - \frac{25}{7} B_3 + \frac{25}{14} B_2 - \frac{1}{2} B_1 + \frac{2}{33} B_0\right)$

$OB_7 = 15\sqrt{3}\left(B_7 - 3.5B_6 + 6.3B_5 - 7B_4 + 5B_3 - \frac{9}{4} B_2 + \frac{7}{12} B_1 - \frac{1}{15} B_0\right)$

$OB_8 = 9\left(B_8 - 4B_7 + \frac{28}{3} B_6 - 14B_5 + 14B_4 - \frac{28}{3} B_3 + 4B_2 - B_1 + \frac{1}{9} B_0\right)$

The above equations can be rewritten in matrix form as

$$OB = MB \tag{7}$$

where

$OB = [OB_0 \quad OB_1 \quad OB_2 \quad OB_3 \quad OB_4 \quad OB_5 \quad OB_6 \quad OB_7 \quad OB_8]^T$

$B = [B_0 \quad B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \quad B_6 \quad B_7 \quad B_8]^T$

$$M = \begin{bmatrix} \sqrt{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{15} & 2\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{13} & -\frac{15\sqrt{13}}{4} & \frac{15\sqrt{13}}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{11} & \frac{21\sqrt{11}}{4} & \frac{39\sqrt{11}}{4} & \frac{-13\sqrt{11}}{2} & 0 & 0 & 0 & 0 & 0 \\ 3 & -\frac{39}{4} & \frac{351}{7} & -\frac{429}{7} & \frac{429}{14} & 0 & 0 & 0 & 0 \\ -\sqrt{7} & \frac{15\sqrt{7}}{2} & \frac{-165}{\sqrt{7}} & \frac{275}{\sqrt{7}} & \frac{-495}{2\sqrt{7}} & \frac{99}{\sqrt{7}} & 0 & 0 & 0 \\ \sqrt{5} & \frac{-33\sqrt{5}}{4} & \frac{825\sqrt{5}}{28} & \frac{-825\sqrt{5}}{14} & \frac{495\sqrt{5}}{7} & -\frac{99\sqrt{5}}{2} & \frac{33\sqrt{5}}{2} & 0 & 0 \\ -\sqrt{3} & \frac{35\sqrt{3}}{4} & \frac{-135\sqrt{3}}{4} & 75\sqrt{3} & -105\sqrt{3} & 54\sqrt{3} & -52.5\sqrt{3} & 15\sqrt{3} & 0 \\ 1 & -9 & 36 & -84 & 126 & -126 & 84 & -36 & 9 \end{bmatrix}$$

6. Operational Matrix of Derivative of OB_8

In this section, we present a general formula for finding the operational matrix of derivative of 8th degree orthonormal Bernstein polynomials .

Let D be an 8×8 operational matrix of the derivative, then

$$\dot{OB} = D_{OB} B \tag{8}$$

The first derivative of n^{th} degree Bernstein basis polynomials can be written as a linear combination of Bernstein basis polynomials of degree n ,for $n = 8$ we have :

$$\dot{B} = D_B B \tag{9}$$

where $\dot{B} = [\dot{B}_0 \quad \dot{B}_1 \quad \dot{B}_2 \quad \dots \quad \dot{B}_7 \quad \dot{B}_8\ ]^T$

$B = [B_0 \quad B_1 \quad B_2 \quad \dots \quad B_7 \quad B_8\ ]^T$

and D_B is the operational matrix of the derivative of 8th degree Bernstein polynomials given by

$$D_B = \begin{bmatrix} -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & -6 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & -4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

By using eq.(9) , we have 9×9 matrix D_{OB} named operational matrix of derivatives of the 8th degree orthonormal Bernstein polynomials and can be obtained as

$$D_{OB} = \begin{bmatrix} -8\sqrt{17} & -\sqrt{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24\sqrt{15} & -11\sqrt{15} & -4\sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 22\sqrt{13} & -\frac{11\sqrt{13}}{4} & -0.1\sqrt{13} & 0.05\sqrt{13} & 0 & 0 & 0 & 0 & 0 \\ -34\sqrt{11} & 35.75\sqrt{11} & 10.5\sqrt{11} & -42.25\sqrt{11} & 26\sqrt{11} & 0 & 0 & 0 & 0 \\ \frac{33}{4} & -\frac{231}{7} & \frac{2145}{7} & \frac{1677}{14} & -\frac{1716}{7} & \frac{2145}{7} & 0 & 0 & 0 \\ -\frac{476}{\sqrt{7}} & \frac{842}{\sqrt{7}} & \frac{-885}{\sqrt{7}} & \frac{192.5}{\sqrt{7}} & \frac{704}{\sqrt{7}} & \frac{1039.5}{\sqrt{7}} & \frac{594}{\sqrt{7}} & 0 & 0 \\ 58\sqrt{5} & -5139.75\sqrt{5} & 219.21\sqrt{5} & -147.32\sqrt{5} & 37.71\sqrt{5} & 205.07\sqrt{5} & -231\sqrt{5} & \frac{231\sqrt{5}}{2} & 0 \\ -62\sqrt{3} & 182.75\sqrt{3} & -297.5\sqrt{3} & 237.75\sqrt{3} & 199.7\sqrt{3} & -259.5\sqrt{3} & 84\sqrt{3} & 277.5\sqrt{3} & 8\sqrt{3} \\ 80 & -305 & 603 & -690 & 168 & 630 & -1020 & 813 & -360 \end{bmatrix}$$

7. Operational Matrix of Integration of OB_8

Let I_{OB} be an 9×9 operational matrix of integration, then

$$\int OB(t)dt = I_{OB}\, B(x) \quad ,0 \le x \le 1 .$$

where the 9×9 matrix I_{OB} is the operational matrix of integration of 8th degree orthonormal Bernstein polynomials on the interval $[0,1]$, and can be written as :

$$I_{OB} = \begin{bmatrix} 0.1049 & 0.2029 & 0.1829 & 0.1689 & 0.1527 & 0.1347 & 0.1138 & 0.0882 & 0.0509 \\ -0.0058 & 0.0926 & 0.1832 & 0.1566 & 0.1440 & 0.1263 & 0.1070 & 0.0823 & 0.0478 \\ 0.000675 & -0.0108 & 0.0802 & 0.1624 & 0.1297 & 0.1191 & 0.0990 & 0.0774 & 0.0444 \\ -0.0124e^{-2} & 0.0020 & -0.0148 & 0.0679 & 0.1404 & 0.1024 & 0.0941 & 0.0697 & 0.0415 \\ 0.0321e^{-3} & -0.0512e^{-2} & 0.0038 & -0.0175 & 0.0556 & 0.1168 & 0.0749 & 0.0681 & 0.0352 \\ -1.0882e^{-5} & 1.7377e^{-4} & -0.0013 & 0.0060 & -0.0188 & 0.0432 & 0.0913 & 0.0476 & 0.0368 \\ 4.5985e^{-6} & -7.3432e^{-5} & 5.4689e^{-4} & -0.0025 & 0.0080 & -0.0183 & 0.0309 & 0.0630 & 0.0206 \\ -2.2667e^{-6} & 3.6196e^{-5} & -2.6958e^{-4} & 0.0012 & -0.0039 & 0.0090 & 0.0152 & 0.0185 & 0.0299 \\ 1.0469e^{-6} & -1.6718e^{-5} & 1.2451e^{-4} & -5.7267e^{-4} & 0.0018 & -0.0042 & 0.0070 & -0.0086 & 0.0062 \end{bmatrix}$$

8. Conclusion

In this study, at first, Orthonormal Bernstein polynomials of degree eight were obtained. Then some formulas were demonstrated the relation between OB_8 and Bernstein polynomials of degree eight .We derived the operational matrices for both derivatives and integration. The given results will be used to solve optimal control problems throughout next work .

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