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# Instability of Axially Loaded Delaminated Composite Cylindrical Shells 

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#### Abstract

This paper is studying local buckling of delaminated cylindrical shells under distributed normal load, where it often causes delamination propagation and structural collapse. The governing differential equations are derived depending on variational principle, the corresponding end conditions and continuity conditions are applied. The associated energy release rate with the delamination is obtained and generalized differential quadrature method is used to solve nonlinear governing equations. Effects of delamination sizes and depths on delamination propagation are discussed.


## Keywords

Differential Quadrature, Delamination, Cylindrical Shells, Axial Load, Buckling

## 1. Introduction

Delamination separates the bonding or continuity between layers and is one the most frequently encountered defects in the structures of composite materials. Delamination arise at the stage of manufacturing, as for example, the adhesion failures and shrinkage cracks.

Despite the rapid development over recent years, problems involving nonlinearity, discontinuity, multiple scale, singularity and irregularity continue to pose challenges in the field of computational science and engineering. Very often, closed form theoretical solutions are unavailable for such complex problems and approximate numerical solutions remain the only recource. Of the various numerical solutions, differential quadrature (DQ) methods have distinguished themselves because of their high accuracy, straightforward implementation and generality in a variety of problems. Not surprisingly, DQ methods have seen phenomenal increase in research interest and experienced significant development in recent years.
S. Rajendran and D. Q. Song [2] discussed finite element modeling of delamination buckling of composite panels
using ANSYS 5.4. The panel was modeled using 8 -node composite shell elements. Zhu Jufen et al. [3] developed a new reference-surface element for analyzing buckling of delaminated composite plates and shells as a new finite element.

Ronald Krueger and T. Kevin O'Brien [4] developed a three-dimensional solid finite element model in vicinity of the delamination front to merge the accuracy of the full threedimensional solution with the computational quality of a plate or shell finite element model. Hayder A. Rasheed and John L. Tassoulas [5] formulated an energy release rate calculation for composite delaminated tubular cross sections and specialized to a finite element model for delamination buckling and growth analysis of long laminated composite tubes considering initial geometric imperfections, large deformations, contact between delamination faces and material degradation.
X. Wanga et al. [6] applied a solving method to conduct research on the non-linear thermal buckling behavior of local delamination near the surface of fiber-reinforced laminated
cylindrical shell.
Zhu Ju-fen et al. [7] presented the reference surface element formulation of a four-node Co quadrilateral membrane-shear-bending element (ZQUA24) and numerical investigations were performed for composite plates and shells with various delamination shapes. Azam Tafreshi [8] dealt with combined double-layer and single-layer shell elements to study the effect of delamination on the global load-carrying capacity of cylindrical shell under axial compressive load. Jinhua Yang and Yiming Fu [9] derived the post-buckling governing equations and the analytical expression of the energy release rates associated with delamination growth in a compression-loaded cylindrical shell by using the variational principle of moving boundary and the Griffith fracture criterion. The finite difference method was used to generate the post-buckling solutions of the delaminated cylindrical shells, and with these solutions, the values of the energy release rates were determined. Jinho Oh et al. [10] developed a new three-node triangular shell element for laminated composite shells with multiple delaminations.

Francesco Tornabene and Erasmo Viola [11] applied the Generalized Differential Quadrature (GDQ) Method to study laminated composite shells of revolution. Yufeng Xing et al. [12] studied the differential quadrature finite element method (DQFEM), as a combination of differential quadrature method (DQM) and standard finite element method (FEM), and formulated one to three - dimensional (1-D to 3-D) element matrices of DQFEM. A Salah et al. [13] used differential quadrature method to solve buckling problem of prismatic and non - prismatic columns.

Multipoint constraint algorithm was inserted in the finite element code to model the delamination by Namita Nanda and S. K. Sahu [14]. Natural frequencies of the delaminated cylindrical (CYL), spherical (SPH) and hyperbolic paraboloid (HYP) shells were obtained by using the shell theories, namely Sanders', Love's, and Donnell's. Based on the higher-order shear deformation theory, Myung-Hyun Noh and Sang-Youl Lee [15] studied the effects that various parameters have on the dynamic stability of delaminated composite skew structures under various periodic in-plane loads. The free vibration analysis of delaminated composite plates was described by Shankar Ganesh et al. [16], based on the first order shear deformation theory (FSDT) and finite
element formulation.

## 2. Generalized Differential Quadrature Method

The basis of differential quadrature method is that the derivatives of a function can be expressed as a weighted linear sum of the function values at discrete points in the domain of the considered variable. Then the derivative of the function can be written as:

$$
\begin{equation*}
\frac{\partial^{m} f\left(x_{i}\right)}{\partial x^{m}}=\sum_{j=1}^{N} C_{i j}^{(m)} f\left(x_{j}\right) \tag{1}
\end{equation*}
$$

Where:
$f\left(x_{j}\right)$ is the function value at grid point $\mathrm{x}_{\mathrm{j}}$.
$C_{i j}^{(m)}$ is the weighting coefficient for the derivative of order ( m ).

Determining the weighting coefficient is the way to link the derivatives in the governing differential equation and the functional values at the grid points.

In order to find simple algebraic formulas for weighting coefficients without restricting the choice of grid points, the test functions are assumed to be Lagrange interpolated polynomials. Thus:

$$
\begin{equation*}
g_{k}(x)=\frac{M(x)}{\left(x-x_{k}\right) \cdot M^{(1)}\left(x_{k}\right)}, k=1,2, \ldots, N \tag{2}
\end{equation*}
$$

Where:

$$
M(x)=\prod_{j=1}^{N}\left(x-x_{j}\right)
$$

And

$$
\begin{equation*}
M^{(1)}(x)=\frac{\partial M(x)}{\partial x}=\prod_{i=1, j=1}^{N}\left(x_{i}-x_{j}\right) \tag{3}
\end{equation*}
$$

For the first order derivative of equation (1) (i.e. $m=1$ ), substituting from equations (2) and (3) into (1) the following relationships can be established:

$$
\begin{equation*}
C_{i j}^{(1)}=\frac{M^{(1)}\left(x_{i}\right)}{\left(x_{i}-x_{j}\right) \cdot M^{(1)}\left(x_{j}\right)}, \text { For } \mathrm{i} \neq \mathrm{j}, i=1,2, \ldots, N \text { and } j=1,2, \ldots, N \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
C_{i i}^{(1)}=\frac{M^{(2)}\left(x_{i}\right)}{2 \cdot M^{(1)}\left(x_{i}\right)} \text {, For } \mathrm{i}=\mathrm{j}, i=1,2, \ldots, N \tag{5}
\end{equation*}
$$

Where, $M^{(2)}\left(x_{i}\right)$ is the second order derivative of $M(x)$.
Equations (4) and (5) are very simple algebraic equations to compute $C_{i j}^{(1)}$ without restriction on choosing sampling grid points. However, the determination of $C_{i i}^{(1)}$ requires the availability of the second order derivative of $M(x)$ which is
more difficult to obtain.
But from equation (1), one can obtain the coefficient $C_{i i}^{(1)}$
as:

$$
\begin{equation*}
C_{i i}^{(1)}=-\sum_{j=1, i \neq j}^{N} C_{i j}^{(1)}, \text { For } i=1,2, \ldots, N \tag{6}
\end{equation*}
$$

Where:

$$
\sum_{j=1}^{N} C_{i j}^{(1)}=0, \text { For } i=1,2, \ldots \ldots, N
$$

Similarly, the weighting coefficients for the second order (i.e. $\mathrm{m}=2$ ) and higher order derivatives can be calculated.

But from the definition of the differential operator, we have

$$
\begin{equation*}
\frac{\partial^{(m)} f}{\partial x^{(m)}}=\frac{\partial}{\partial x}\left(\frac{\partial^{(m-1)} f}{\partial x^{(m-1)}}\right)=\frac{\partial^{(m-1)}}{\partial x^{(m-1)}}\left(\frac{\partial f}{\partial x}\right) \tag{7}
\end{equation*}
$$

Let $\left[\mathrm{A}^{(\mathrm{m}-1)}\right]$, $\left[\mathrm{A}^{(\mathrm{m})}\right]$ be the weighting coefficient matrices of the ( $\mathrm{m}-1$ )th and m th order derivatives respectively. Then the application of differential quadrature approximation to equation (7) results in the following recurrence relationship:

$$
\begin{equation*}
\left[A^{(m)}\right]=\left[A^{(1)}\right]\left[A^{(m-1)}\right]=\left[A^{(m-1)}\right]\left[A^{(1)}\right] \tag{8}
\end{equation*}
$$

To summarize, the equation (8) and the formulations for the coefficients of first derivatives (4), (6) constitute complete formula for the determination of the weighting coefficients from the first to as high as (m-1)th order derivatives.

For simplification, the grid points are chosen equally spaced, as shown in Figure 1. These are given in X- direction as:

$$
\begin{equation*}
X_{i}=\frac{i-1}{N_{x}-1} ; i=1,2, \ldots \ldots, N_{x} \tag{9}
\end{equation*}
$$



Figure 1. Equally spaced grid points.
But in this paper, the two-dimensional treatment of the cylindrical shell will be introduced, so the number of grid points in y direction will be also equally spaced but less than the x direction by 4 points, due to implementation of boundary conditions in $x$-direction as will be shown later, to can create a square weighting coefficient matrices.

## 3. Formulation of the Problem

Consider a cylindrical shell with throughout circumference delamination having mid surface radius $R$, thickness $h$, length $L$, and mass density $\rho$ under the action of uniform axial load $\bar{N}$, and the shell is referred to the coordinate system $x, y, z$ as shown in Figures 2 and 3.


Figure 2. Geometry of Cylindrical Shell with Delamination.


Figure 3. Longitudinal Cross Section within Shell Surface Generator.

The delaminated length of the shell is $\beta L, \beta$ is the delamination length parameter, $Z^{*}$ is the distance measured from the shell mid-surface to the delamination interface and $\ell$ represents the delamination position measured from the left end of the shell.

In order to investigate delamination growth, delaminated cylindrical shell is divided into four regions which are respectively denoted by $\Omega_{i}(i=1,2,3,4)$. Here, 2 and 3 represent delaminated segments, 1 and 4 represent intact segments .The coordinate $x$ for each region is measured from the left end. The thicknesses of region 2,3 are $h_{2}, h_{3}$ respectively and it is obvious that $h_{2}+h_{3}=h$. Delamination growth for laminated cylindrical shell has two boundaries denoted as $C_{j}(j=1,2)$. Boundaries on both ends of the shell are denoted by $C_{o}$.

Supposing that $\bar{u}_{i}, \bar{v}_{i}, \bar{w}_{i}$ denote axial, circumferential and radial displacements of any point on the region $\Omega_{i}$, and the corresponding displacement components of middle surface are $u_{i}, v_{i}, w_{i}$ then the displacement components are given by:

$$
\begin{gather*}
\bar{u}_{i}=u_{i}-z w_{i, x}(x, y) \\
\bar{v}_{i}=v_{i}-z w_{i, y}(x, y)  \tag{10}\\
\bar{w}_{i}=w_{i}
\end{gather*}
$$

Assuming $\overline{\mathcal{E}}_{\text {ix }}, \overline{\mathcal{E}}_{\text {iy }}$ and $\overline{\mathcal{E}}_{\text {ixy }}$ denote strains of any point in region $\Omega_{i}$, the nonlinear strain displacement relation may be written as:

$$
\begin{align*}
& \overline{\mathcal{E}}_{\mathrm{ix}}=\mathcal{E}_{\mathrm{ix}}+\mathrm{z} \mathrm{\kappa} \kappa_{\mathrm{ix}} \\
& \overline{\mathcal{E}}_{\mathrm{iy}}=\mathcal{E}_{\mathrm{ix}}+\mathrm{z} \mathrm{i}_{\mathrm{iy}}  \tag{11}\\
& \overline{\mathcal{E}}_{\mathrm{ixy}}=\mathcal{E}_{\mathrm{ixy}}+\mathrm{z} \kappa_{\mathrm{ixy}}
\end{align*}
$$

Where $\mathcal{E}_{\mathrm{ix}}, \mathcal{E}_{\mathrm{ix}}$ and $\mathcal{E}_{\mathrm{ixy}}$ are the strains of the middle surface and $\kappa_{\mathrm{ix}}, \kappa_{\mathrm{i} y}$ and $\kappa_{\mathrm{ixy}}$ are the change of values of curvature of the middle surface of radius $R_{i}$, and:

$$
\begin{gather*}
\mathcal{E}_{\mathrm{ix}}=\mathrm{u}_{\mathrm{i}, \mathrm{x}}+\frac{1}{2} w_{i, x}^{2} \\
\mathcal{E}_{\mathrm{iy}}=v_{\mathrm{i}, \mathrm{y}}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}  \tag{12}\\
\mathcal{E}_{\mathrm{ixy}}=\mathrm{u}_{\mathrm{i}, \mathrm{y}}+\mathrm{v}_{\mathrm{i}, \mathrm{x}}+w_{i, x} w_{i, y} \\
\kappa_{\mathrm{ix}}==w_{i, x x} \\
\kappa_{\mathrm{iyy}}==w_{i, y y}  \tag{13}\\
\kappa_{\mathrm{ixy}}=2 w_{i, x y}
\end{gather*}
$$

According to classical theory of shells, the membrane stress resultants $N_{i x}, N_{i y}, N_{i x y}$ and stress couples $M_{i x}, M_{i y}, M_{i x y}$ can be written as:

$$
\left[\begin{array}{l}
{\left[N_{i}\right]}  \tag{14}\\
{\left[M_{i}\right]}
\end{array}\right]=\left[\begin{array}{ll}
{\left[A_{i j}^{(i)}\right]} & {\left[B_{i j}^{(i)}\right]} \\
{\left[B_{i j}^{(i)}\right]} & {\left[D_{i j}^{(i)}\right]}
\end{array}\right]\left[\begin{array}{l}
{\left[\begin{array}{l}
\left.\varepsilon_{i}\right] \\
{\left[\kappa_{i}\right]}
\end{array}\right], ~}
\end{array}\right]
$$

In which:

$$
\begin{gather*}
{\left[N_{i}\right]=\left[\begin{array}{c}
N_{i x} \\
N_{i y} \\
N_{i x y}
\end{array}\right],\left[M_{i}\right]=\left[\begin{array}{c}
M_{i x} \\
M_{i y} \\
M_{i x y}
\end{array}\right],\left[\mathcal{E}_{i}\right]=\left[\begin{array}{c}
\mathcal{E}_{i x} \\
\mathcal{E}_{i y} \\
\mathcal{E}_{i x y}
\end{array}\right],\left[\kappa_{i}\right]=\left[\begin{array}{c}
\kappa_{i x} \\
\kappa_{i y} \\
\kappa_{i x y}
\end{array}\right]}  \tag{15}\\
\left(A_{i j}^{(i)}, B_{i j}^{(i)}, D_{i j}^{(i)}\right)=\int_{h_{i} / 2}^{h_{i} / 2}{\overline{Q_{\imath j}}}^{(k)}\left(1, z, z^{2}\right) d z(i, j=1,2,6) \tag{16}
\end{gather*}
$$

Where $A_{i j}^{(i)}, B_{i j}^{(i)}, D_{i j}^{(i)}$ are extension, coupling and bending rigidity, respectively, and ${\overline{Q_{l j}}}^{(k)}$ is elastic constant of the $k^{t h}$ layer. Assuming that the generalized forces acting on the boundary $C_{o}$ are:

$$
\begin{equation*}
\left[P_{1}^{(\gamma)}, P_{2}^{(\gamma)}, P_{3}^{(\gamma)}, P_{4}^{(\gamma)}, P_{5}^{(\gamma)}\right]=\left[\bar{N}_{x}^{(\gamma)}, \bar{N}_{y}^{(\gamma)}, \bar{M}_{x}^{(\gamma)}, \bar{M}_{y}^{(\gamma)}, \bar{Q}^{(\gamma)}\right] \tag{17}
\end{equation*}
$$

And the generalized displacements of each region are:

$$
\begin{equation*}
\left[u_{1}^{(i)}, u_{2}^{(i)}, u_{3}^{(i)}, u_{4}^{(i)}, u_{5}^{(i)}\right]=\left[u_{i}, v_{i}, w_{i, x}, w_{i, y}, w_{i}\right](\mathrm{i}=1,2,3,4) \tag{18}
\end{equation*}
$$

Then the total potential energy of delaminated cylindrical shell can be written as:

$$
\begin{equation*}
\pi=\sum_{i=1}^{4} \iiint_{\Omega_{i}} U_{i} d x d y d z-\sum_{\gamma=1,4} \int_{S_{\gamma}} \sum_{j=1}^{5} P_{j}^{(\gamma)} u_{j}^{(\gamma)} d S_{\gamma} \tag{19}
\end{equation*}
$$

Where $U_{i}$ is the strain energy density relative to region $\Omega_{i}$.
Using equation (14) and noticing that the problem of delamination growth involves variation of moving boundary, then the variation of total potential energy is:

$$
\delta \pi=\sum_{i=1}^{4} \iint_{\Omega_{i}+\delta \Omega_{i}} \frac{1}{2}\left\{\begin{array}{l}
{\left[\varepsilon^{(i)}+\delta \varepsilon^{(i)}\right]}  \tag{20}\\
{\left[\kappa^{(i)}+\delta \kappa^{(i)}\right]}
\end{array}\right\}\left[\begin{array}{ll}
A^{(i)} & B^{(i)} \\
B^{(i)} & D^{(i)}
\end{array}\right]\left\{\begin{array}{l}
{\left[\varepsilon^{(i)}+\delta \varepsilon^{(i)}\right]} \\
{\left[\kappa^{(i)}+\delta \kappa^{(i)}\right]}
\end{array}\right\} d x-\sum_{\gamma=1,4} \int_{S} \sum_{j=1}^{5} P_{j}^{(\gamma)}\left(u_{j}^{(\gamma)}+\delta u_{j}^{(\gamma)}\right) d s_{\gamma}-\pi
$$

Moreover, it is assumed that the shell is symmetrically laminated and all regions are still symmetric with respect to their each mid surface after delamination. Then $\left[B^{(i)}\right]=0$, and then the above equation can be written as:

$$
\begin{gather*}
\delta \pi=\sum_{i=1}^{4} \iint_{\Omega_{i}}\left\{\left[\varepsilon^{i}\right]^{T}\left[A^{i}\right]\left[\delta \varepsilon^{i}\right]+\left[\kappa^{i}\right]^{T}\left[D^{i}\right]\left[\delta \kappa^{i}\right]\right\} d x d y- \\
\sum_{\gamma=1,4} \int_{S} \sum_{j=1}^{5} P_{j}^{(\gamma)} \delta u_{j}^{(\gamma)} d s_{\gamma}+\frac{1}{2} \sum_{i=1}^{4} \iint_{\delta \Omega_{i}}\left\{\left[\varepsilon^{i}\right]^{T}\left[A^{i}\right]\left[\varepsilon^{i}\right]+\left[\kappa^{i}\right]^{T}\left[D^{i}\right]\left[\kappa^{i}\right]\right\} d x d y \tag{21}
\end{gather*}
$$

The third term in equation (21) can be given in the following form:

$$
\begin{equation*}
\sum_{i=1}^{4} \iint_{\delta \Omega_{i}}\left\{\left[\varepsilon^{i}\right]^{T}\left[A^{i}\right]\left[\varepsilon^{i}\right]+\left[\kappa^{i}\right]^{T}\left[D^{i}\right]\left[\kappa^{i}\right]\right\} d x d y=\sum_{i=1}^{4} \oint_{C_{j}}\left\{\left[\varepsilon^{i}\right]^{T}\left[A^{i}\right]\left[\varepsilon^{i}\right]+\left[\kappa^{i}\right]^{T}\left[D^{i}\right]\left[\kappa^{i}\right]\right\} \delta n_{i} d C_{j} \tag{22}
\end{equation*}
$$

Where $n$ the normal direction of delamination growth is, $\delta n_{i}$ is the change due to delamination propagation at the variable boundary $d C_{j}$.

Using equations (13) to (16), equation (22) changes to:

$$
\begin{gather*}
\delta \pi=\sum_{i=1}^{4} \iint_{\Omega_{i}}\left\{N_{x i} \delta\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i} \delta\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i} \delta\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+M_{x i} \delta\left(-w_{i, x x}\right)+\right. \\
\left.M_{y i} \delta\left(-w_{i, y y}\right)+M_{x y i} \delta\left(-2 w_{i, x y}\right)\right\} d x d y+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+\right.\right. \\
\left.\left.w_{i, x} w_{i, y}\right)+M_{x i}\left(-w_{i, x x}\right)+M_{y i}+M_{x y i}\left(-2 w_{i, x y}\right)\right\} \delta n_{i} d C_{j} \tag{23}
\end{gather*}
$$

Using differential and integral calculus, one can deduce that:

$$
\begin{gather*}
\delta \pi=\sum_{i=1}^{4} \iint_{\Omega_{i}}\left\{\left[-\frac{\partial N_{x i}}{\partial x}-\frac{\partial N_{x y i}}{\partial y}\right] \delta u_{i}+\left[-\frac{\partial N_{x y i}}{\partial x}-\frac{\partial N_{y i}}{\partial y}\right] \delta v_{i}+\left[-\frac{\partial M_{x i, x}}{\partial x}-\frac{\partial M_{y i, y}}{\partial y}-\frac{\partial M_{x y i, x}}{\partial y}-\frac{\partial M_{x y i, y}}{\partial x}-\frac{\partial\left(N_{x i} w_{i, x}\right)}{\partial x}-\frac{\partial\left(N_{y i} w_{i, y}\right)}{\partial y}-\right.\right. \\
\left.\left.\frac{\partial\left(N_{x y i} w_{i, y)}\right.}{\partial x}-\frac{\partial\left(N_{x y i} w_{i, x}\right)}{\partial y}-\frac{N_{y i}}{R_{i}}\right] \delta w_{i}\right\} d x d y+\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i} \delta u_{i}+N_{x i} w_{i, x} \delta w_{i}+N_{x y i} \delta v_{i}+N_{x y i} w_{i, y} \delta w_{i}-M_{x i} \delta w_{i, x}+\right. \\
\left.M_{x i, x} \delta w_{i}-M_{x y i} \delta w_{i, y}+M_{x y i, y} \delta w_{i}\right\} d y+\left\{N_{y i} \delta v_{i}+N_{y i} w_{i, y} \delta w_{i}+N_{x y i} \delta u_{i}+N_{x y i} w_{i, x} \delta w_{i}-M_{y i} \delta w_{i, y}+M_{y i, y} \delta w_{i}-\right. \\
\left.M_{x y i} \delta w_{i, x}+M_{x y i, x} \delta w_{i}\right\} d x-\sum_{y=1,4} \int_{S} \sum_{j=1}^{5} P_{j}^{(\gamma)} \delta u_{j}^{(\gamma)} d S+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+\right. \\
\left.N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+M_{x i}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+M_{x y i}\left(-2 w_{i, x y}\right)\right\} \delta n_{i} d C_{j} \tag{24}
\end{gather*}
$$

For delaminated cylindrical shell, the normal direction $n_{i}$ of delamination growth is consistent with the axial direction x . Therefore the above equation can be written as:

$$
\begin{array}{r}
\delta \pi=\sum_{i=1}^{4} \iint_{\Omega_{i}}\left\{\left[-\frac{\partial N_{x i}}{\partial x}-\frac{\partial N_{x y i}}{\partial y}\right] \delta u_{i}+\left[-\frac{\partial N_{x y i}}{\partial x}-\frac{\partial N_{y i}}{\partial y}\right] \delta v_{i}+\left[-\frac{\partial M_{x i, x}}{\partial x}-\frac{\partial M_{y i, y}}{\partial y}-\frac{\partial M_{x y i, x}}{\partial y}-\frac{\partial M_{x y i, y}}{\partial x}-\frac{\partial\left(N_{x i} w_{i, x}\right)}{\partial x}-\frac{\partial\left(N_{y i} w_{i, y}\right)}{\partial y}-\right.\right. \\
\left.\left.\frac{\partial\left(N_{x y i} w_{i, y}\right)}{\partial x}-\frac{\partial\left(N_{x y i} w_{i, x}\right)}{\partial y}-\frac{N_{y i}}{R_{i}}\right] \delta w_{i}\right\} d x d y+\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i} \delta u_{i}+N_{x i} w_{i, x} \delta w_{i}+N_{x y i} \delta v_{i}+N_{x y i} w_{i, y} \delta w_{i}-M_{x i} \delta w_{i, x}+\right. \\
\left.M_{x i, x} \delta w_{i}-M_{x y i} \delta w_{i, y}+M_{x y i, y} \delta w_{i}\right\} d C_{j}-\sum_{\gamma=1,4} \int_{S} \sum_{j=1}^{5} P_{j}^{(\gamma)} \delta u_{j}^{(\gamma)} d S+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\right.\right. \\
\left.\left.\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+M_{x i}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+M_{x y i}\left(-2 w_{i, x y}\right)\right\} \delta n_{i} d C_{j} \tag{25}
\end{array}
$$

As the variation of $u_{i}, v_{i}, w_{i}$ is carried out on variable boundary $\mathrm{C}_{\mathrm{j}}$, then we have:

$$
\begin{gather*}
\left.\delta u_{i}\right|_{C_{j}}=\delta\left(\left.u_{i}\right|_{C_{j}}\right)-\left.\frac{\partial u_{i}}{\partial n}\right|_{C_{j}} . \delta n_{i},\left.\delta v_{i}\right|_{C_{j}}=\delta\left(\left.v_{i}\right|_{C_{j}}\right)-\left.\frac{\partial v_{i}}{\partial n}\right|_{C_{j}} . \delta n_{i} \\
\left.\delta w_{i}\right|_{C_{j}}=\delta\left(\left.w_{i}\right|_{C_{j}}\right)-\left.\frac{\partial w_{i}}{\partial n}\right|_{C_{j}} . \delta n_{i},\left.\delta w_{i, x}\right|_{C_{j}}=\delta\left(\left.w_{i, x}\right|_{C_{j}}\right)-\left.\frac{\partial w_{i, x}}{\partial n}\right|_{C_{j}} . \delta n_{i} \\
\left.\delta w_{i, y}\right|_{C_{j}}=\delta\left(\left.w_{i, y}\right|_{C_{j}}\right)-\left.\frac{\partial w_{i, y}}{\partial n}\right|_{C_{j}} . \delta n_{i} \tag{26}
\end{gather*}
$$

Using equation (25), equation (26) can be written as:

$$
\begin{array}{r}
\delta \pi=\sum_{i=1}^{4} \iint_{\Omega_{i}}\left\{\left[-\frac{\partial N_{x i}}{\partial x}-\frac{\partial N_{x y i}}{\partial y}\right] \delta u_{i}+\left[-\frac{\partial N_{x y i}}{\partial x}-\frac{\partial N_{y i}}{\partial y}\right] \delta v_{i}+\left[-\frac{\partial M_{x i, x}}{\partial x}-\frac{\partial M_{y i, y}}{\partial y}-\frac{\partial M_{x y i, x}}{\partial y}-\frac{\partial M_{x y i, y}}{\partial x}-\frac{\partial\left(N_{x i} w_{i, x}\right)}{\partial x}-\frac{\partial\left(N_{y i} w_{i, y}\right)}{\partial y}-\right.\right. \\
\left.\left.\frac{\partial\left(N_{x y i} w_{i, y}\right)}{\partial x}-\frac{\partial\left(N_{x y i} w_{i, x}\right)}{\partial y}-\frac{N_{y i}}{R_{i}}\right] \delta w_{i}\right\} d x d y+\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i} \delta\left(\left.u_{i}\right|_{C_{j}}\right)+N_{x y i} \delta\left(\left.v_{i}\right|_{C_{j}}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+M_{x i, x}+\right.\right.
\end{array}
$$

$$
\begin{gather*}
\left.\left.M_{x y i, y}\right) \delta\left(\left.w_{i}\right|_{c_{j}}\right)-M_{x i} \delta\left(\left.w_{i, x}\right|_{C_{j}}\right)-M_{x y i} \delta\left(\left.w_{i, y}\right|_{C_{j}}\right)\right\} d C_{j}-\sum_{\gamma=1,4} \int_{S} \sum_{j=1}^{5} P_{j}^{(\gamma)} \delta u_{j}^{(\gamma)} d S+\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(-\left.\frac{\partial u_{i}}{\partial n}\right|_{C_{j}}\right)+\right. \\
\left.N_{x y i}\left(-\left.\frac{\partial v_{i}}{\partial n}\right|_{C_{j}}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+M_{x i, x}+M_{x y i, y}\right)\left(-\left.\frac{\partial w_{i}}{\partial n}\right|_{c_{j}}\right)+M_{x i}\left(\left.\frac{\partial w_{i, x}}{\partial n}\right|_{C_{j}}\right)+M_{x y i}\left(\left.\frac{\partial w_{i, y}}{\partial n}\right|_{C_{j}}\right)\right\} \delta n_{i} d C_{j} \\
+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+M_{x i}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+\right. \\
\left.M_{x y i}\left(-2 w_{i, x y}\right)\right\} \delta n_{i} d C_{j} \tag{27}
\end{gather*}
$$

Equation (27) can be written as the following two parts, that is:

$$
\begin{equation*}
\delta \pi=\delta \pi_{1}+\delta \pi_{2} \tag{28}
\end{equation*}
$$

Where:

$$
\begin{array}{r}
\delta \pi_{1}=\sum_{i=1}^{4} \iint_{\Omega_{i}}\left\{\left[-\frac{\partial N_{x i}}{\partial x}-\frac{\partial N_{x y i}}{\partial y}\right] \delta u_{i}+\left[-\frac{\partial N_{x y i}}{\partial x}-\frac{\partial N_{y i}}{\partial y}\right] \delta v_{i}+\left[-\frac{\partial M_{x i x}}{\partial x}-\frac{\partial M_{y i, y}}{\partial y}-\frac{\partial M_{x y i, x}}{\partial y}-\frac{\partial M_{x y i, y}}{\partial x}-\frac{\partial\left(N_{x i} w_{i, x}\right)}{\partial x}-\right.\right. \\
\left.\left.\frac{\partial\left(N_{y i} w_{i, y}\right)}{\partial y}-\frac{\partial\left(N_{x y i} w_{i, y}\right)}{\partial x}-\frac{\partial\left(N_{x y i} w_{i, x}\right.}{\partial y}-\frac{N_{y i}}{R_{i}}\right] \delta w_{i}\right\} d x d y+\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i} \delta\left(u_{i} \mid c_{c_{j}}\right)+N_{x y i} \delta\left(\left.v_{i}\right|_{c_{j}}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+\right.\right. \\
\left.\left.M_{x i, x}+M_{x y i, y}\right) \delta\left(\left.w_{i}\right|_{c_{j}}\right)-M_{x i} \delta\left(\left.w_{i, x}\right|_{C_{j}}\right)-M_{x y i} \delta\left(\left.w_{i, y}\right|_{C_{j}}\right)\right\} d C_{j}-\sum_{r=1,4} \int_{S} \sum_{j=1}^{5} P_{j}^{(\gamma)} \delta u_{j}^{(\gamma)} d S \\
\delta \pi_{2}=\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(-\left.\frac{\partial u_{i}}{\partial n}\right|_{C_{j}}\right)+N_{x y i}\left(-\left.\frac{\partial v_{i}}{\partial n}\right|_{C_{j}}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+M_{x i, x}+M_{x y i, y}\right)\left(-\left.\frac{\partial w_{i}}{\partial n}\right|_{c_{j}}\right)+M_{x i}\left(\left.\frac{\partial w_{i, x}}{\partial n}\right|_{C_{j}}\right)+\right. \\
\left.M_{x y i}\left(\left.\frac{\partial w_{i, y}}{\partial n}\right|_{C_{j}}\right)\right\} \delta n_{i} d C_{j}+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+\right. \\
\left.M_{x i, y}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+M_{x y i}\left(-2 w_{i, x y}\right)\right\} \delta n_{i} d C_{j} \tag{30}
\end{array}
$$

For the delaminated cylindrical shell, the normal direction $n$ of delamination growth is consistent with the axial direction $x$. Therefore, equation (30) can also be written as follows:

$$
\begin{gather*}
\delta \pi_{2}=\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(-u_{i, x}\right)+N_{x y i}\left(-v_{i, x}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+M_{x i, x}+M_{x y i, y}\right)\left(-\left.\frac{\partial w_{i}}{\partial x}\right|_{C_{j}}\right)+M_{x i}\left(\left.\frac{\partial w_{i, x}}{\partial x}\right|_{C_{j}}\right)+\right. \\
\left.M_{x y i}\left(\left.\frac{\partial w_{i, y}}{\partial x}\right|_{C_{j}}\right)\right\} \delta n_{i} d C_{j}+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+\right. \\
\left.M_{x i}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+M_{x y i}\left(-2 w_{i, x y y}\right)\right\} \delta n_{i} d C_{j} \tag{31}
\end{gather*}
$$

The displacements of laminated cylindrical shell must change after imaginary growth $\delta n$ occurs along delamination front. At the same time, the changeable area in the region of integration is that is $\delta \mathrm{A}_{\mathrm{i}}=\int_{C_{j}} \delta n_{i} d C_{j}$. Thus, $\delta \pi_{2}$ is the variation of potential energy due to the area alteration of each region.
$\delta \pi_{1}$ is the variation of the potential energy due to the virtual displacement of laminated cylindrical shell while imaginary growth does not occur (i. e the delamination
growth is immovable). When the laminated cylindrical shell is in state of equilibrium, according to the principle of virtual displacement, we have:

$$
\begin{equation*}
\delta \pi_{1}=0 \tag{32}
\end{equation*}
$$

From equation (32), the post buckling governing equations for each region are:

$$
\begin{gather*}
A_{11}^{(i)} u_{i, x x}+\left(A_{12}^{(i)}+A_{66}^{(i)}\right) v_{i, x y}+\left(A_{12}^{(i)}+A_{66}^{(i)}\right) w_{i, y} w_{i, x y}+A_{11}^{(i)} w_{i, x} w_{i, x x}+A_{66}^{(i)} w_{i, x} w_{i, y y}+A_{66}^{(i)} u_{i, y y}-A_{12}^{(i)} \frac{1}{R} w_{i, x}=0 \\
\left(A_{12}^{(i)}+A_{66}^{(i)}\right) u_{i, x y}+A_{66}^{(i)} v_{i, x x}+A_{22}^{(i)} v_{i, y y}+\left(A_{12}^{(i)}+A_{66}^{(i)}\right) w_{i, x} w_{i, x y}+A_{66}^{(i)} w_{i, y} w_{i, x x}+A_{22}^{(i)} w_{i, y} w_{i, y y}-A_{22}^{(i)} \frac{1}{R} w_{i, y}=0 \\
-\left(D_{11}^{(i)} w_{i, x x x x x}+\left(2 D_{12}^{(i)}+4 D_{66}^{(i)}\right) w_{i, x x y y}+D_{22}^{(i)} w_{i, y y y y}\right)+w_{i, x x}\left(A_{11}^{(i)} u_{i, x}+\frac{1}{2} A_{11}^{(i)} w_{i, x}^{2}+A_{12}^{(i)} v_{i, y}-A_{12}^{(i)} \frac{1}{R} w_{i}+\right. \\
\left.\frac{1}{2} A_{12}^{(i)} w_{i, y}^{2}\right)+2 A_{66}^{(i)}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right) w_{i, x y}+\left(A_{12}^{(i)} u_{i, x}+\frac{1}{2} A_{12}^{(i)} w_{i, x}^{2}+A_{22}^{(i)} v_{i, y}-A_{22}^{(i)} \frac{1}{R} w_{i}+\frac{1}{2} A_{22}^{(i)} w_{i, y}^{2}\right)\left(w_{i, y y}+\right. \\
\left.\frac{1}{R_{i}}\right)=0 \tag{33}
\end{gather*}
$$

### 3.1. Continuity Conditions

The continuity conditions of displacement can be described as follows:

$$
\begin{gather*}
u_{2}(0, y)=u_{1}\left(L_{1}, y\right)+d_{2} w_{1, x}\left(L_{1}, y\right), v_{2}(0, y)=v_{1}\left(L_{1}, y\right)+d_{2} w_{1, y}\left(L_{1}, y\right) \\
u_{3}(0, y)=u_{1}\left(L_{1}, y\right)-d_{3} w_{1, x}\left(L_{1}, y\right), v_{3}(0, y)=v_{1}\left(L_{1}, y\right)-d_{3} w_{1, y}\left(L_{1}, y\right) \\
u_{2}\left(L_{2}, y\right)=u_{4}(0, y)+d_{2} w_{4, x}(0, y), v_{2}\left(L_{2}, y\right)=v_{4}(0, y)+d_{2} w_{4, y}(0, y) \\
u_{3}\left(L_{3}, y\right)=u_{4}(0, y)-d_{3} w_{4, x}(0, y), v_{3}\left(L_{3}, y\right)=v_{4}(0, y)-d_{3} w_{4, y}(0, y) \\
w_{1}\left(L_{1}, y\right)=w_{2}(0, y)=w_{3}(0, y), w_{1, x}\left(L_{1}, y\right)=w_{2, x}(0, y)=w_{3, x}(0, y) \\
w_{4}(0, y)=w_{2}\left(L_{2}, y\right)=w_{3}\left(L_{3}, y\right), w_{4, x}(0, y)=w_{2, x}\left(L_{2}, y\right)=w_{3, x}\left(L_{3}, y\right) \tag{34}
\end{gather*}
$$

Where:

$$
d_{i}=\frac{h}{2}-\frac{h_{i}}{2}
$$

### 3.2. Equilibrium Conditions

The equilibrium conditions of moments and forces which must be satisfied at both ends of delamination are:

$$
\begin{gather*}
N_{1_{x x}}\left(L_{1}, y\right)=N_{2_{x x}}(0, y)+N_{3_{x x}}(0, y), N_{4 x x}(0, y)=N_{2_{x x}}\left(L_{2}, y\right)+N_{3_{x x}}\left(L_{3}, y\right) \\
N_{1_{x y}}\left(L_{1}, y\right)=N_{2_{x y}}(0, y)+N_{3_{x y}}(0, y), N_{4 x y}(0, y)=N_{2_{x y}}\left(L_{2}, y\right)+N_{3 x y}\left(L_{3}, y\right) \\
M_{1 x}\left(L_{1}, y\right)=M_{2 x}(0, y)-d_{2} N_{2 x x}(0, y)+M_{3 x}(0, y)+d_{3} N_{3_{x x}}(0, y) \\
M_{4 x}(0, y)=M_{2 x}\left(L_{2}, y\right)-d_{2} N_{2_{x x}}\left(L_{2}, y\right)+M_{3 x}\left(L_{3}, y\right)+d_{3} N_{3 x x}\left(L_{3}, y\right) \\
Q_{1 x}\left(L_{1}, y\right)=Q_{2 x}(0, y)+Q_{3 x}(0, y), Q_{4 x}(0, y)=Q_{2 x}\left(L_{2}, y\right)+Q_{3 x}\left(L_{3}, y\right) \tag{35}
\end{gather*}
$$

### 3.3. Boundary Conditions

The boundary conditions for both ends may be:

### 3.3.1. For Clamped Ends

$$
\begin{gather*}
w_{1}(0, y)=0, N_{1_{x x}}(0, y)=-\bar{N}, N_{1_{x y}}(0, y)=0, w_{1, x}(0, y)=0 \\
w_{4}\left(L_{4}, y\right)=0, N_{4_{x x}}\left(L_{4}, y\right)=-\bar{N}, N_{4_{x y}}\left(L_{4}, y\right)=0, w_{4, x}\left(L_{4}, y\right)=0 \tag{36}
\end{gather*}
$$

### 3.3.2. For Simply Supported Ends

$$
\begin{gather*}
w_{1}(0, y)=0, N_{1_{x x}}(0, y)=-\bar{N}, N_{1_{x y}}(0, y)=0, w_{1, x x}(0, y)=0 \\
w_{4}\left(L_{4}, y\right)=0, N_{4_{x x}}\left(L_{4}, y\right)=-\bar{N}, N_{4_{x y}}\left(L_{4}, y\right)=0, w_{4, x x}\left(L_{4}, y\right)=0 \tag{37}
\end{gather*}
$$

## 4. Energy Release Rate

Two approaches for predicting delamination initiation currently exist: the strain energy release rate approach and the strength of material approach. The strain energy release rate is based on the fracture mechanics concept that sufficient strain energy must be available to create a new surface where delamination initiates. Here, the energy release rate approach will be followed.

When an imaginary growth $\delta n$ occurs, the variation of the total potential energy is:

$$
\begin{array}{r}
\delta \pi=\delta \pi_{2}=\sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(-u_{i, x}\right)+N_{x y i}\left(-v_{i, x}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+M_{x i, x}+M_{x y i, y}\right)\left(-\left.\frac{\partial w_{i}}{\partial x}\right|_{C_{j}}\right)+M_{x i}\left(\left.\frac{\partial w_{i, x}}{\partial x}\right|_{C_{j}}\right)+\right. \\
\left.M_{x y i}\left(\left.\frac{\partial w_{i, y}}{\partial x}\right|_{C_{j}}\right)\right\} \delta n_{i} d C_{j}+\frac{1}{2} \sum_{i=1}^{4} \oint_{C_{j}}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+\right.
\end{array}
$$

$$
\begin{equation*}
\left.M_{x i}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+M_{x y i}\left(-2 w_{i, x y}\right)\right\} \delta n_{i} d C_{j} \tag{38}
\end{equation*}
$$

Let

$$
\begin{gather*}
G_{i}=\left\{N_{x i}\left(-u_{i, x}\right)+N_{x y i}\left(-v_{i, x}\right)+\left(N_{x i} w_{i, x}+N_{x y i} w_{i, y}+M_{x i, x}+M_{x y i, y}\right)\left(-\left.\frac{\partial w_{i}}{\partial x}\right|_{C_{j}}\right)+M_{x i}\left(\left.\frac{\partial w_{i, x}}{\partial x}\right|_{C_{j}}\right)+M_{x y i}\left(\left.\frac{\partial w_{i, y}}{\partial x}\right|_{C_{j}}\right)\right\}+ \\
\frac{1}{2}\left\{N_{x i}\left(u_{i, x}+\frac{1}{2} w_{i, x}^{2}\right)+N_{y i}\left(v_{i, y}-\frac{w_{i}}{R_{i}}+\frac{1}{2} w_{i, y}^{2}\right)+N_{x y i}\left(u_{i, y}+v_{i, x}+w_{i, x} w_{i, y}\right)+M_{x i}\left(-w_{i, x x}\right)+M_{y i}\left(-w_{i, y y}\right)+\right. \\
\left.M_{x y i}\left(-2 w_{i, x y}\right)\right\} \tag{39}
\end{gather*}
$$

Then

$$
\begin{equation*}
\delta \pi=\sum_{i=1}^{4} \oint_{C_{j}} G_{i} \delta n_{i} d C_{j} \tag{40}
\end{equation*}
$$

The essence of delamination growth is that the delamination boundary continually moves and so the formulas of energy release rate can be found according to Griffith criterion of crack growth. The area variation of the delamination region is denoted by $\delta A$. According to the energy conservation principle, the work done by the external loads is the summation of the elastic strain energy and the energy spent on the delamination growth, that is:

$$
\begin{equation*}
\delta(\pi+\Gamma)=0 \tag{41}
\end{equation*}
$$

Where $\Gamma$ represents the energy spent on the delamination growth. According to Griffith criterion, the energy release rate can be expressed as:

$$
\begin{equation*}
G=-\lim _{\delta A \rightarrow 0} \frac{\delta \pi}{\delta A} \tag{42}
\end{equation*}
$$

So the average energy release rate $G_{a}$ of delamination growth is:

$$
\begin{equation*}
G_{a}=-\frac{\delta \pi}{\delta A}=-\frac{\sum_{i=1}^{4} \oint_{C_{j}} G_{i} \delta n_{i} d C_{j}}{\int_{C_{j}} \delta n d C_{j}} \tag{43}
\end{equation*}
$$

For partial delamination growth on delamination boundary, i. e. non - within thickness delamination, $\delta n$ is greater than zero on certain part of the boundary $\Delta C_{j}$ and equal to zero on residual part of the boundary $C_{j}-\Delta C_{j}$. Supposing $\Delta C_{j}$ is a small segment including a given point and letting $\Delta C_{j}$ infinitely minish to approach the point, then the energy release rate of any point can be given as:

$$
\begin{gathered}
U_{i, X X}+\left(\frac{1+v}{2}\right) V_{i, X Y}+\left(\frac{1-v}{2}\right) U_{i, Y Y}=-\left(\frac{1+v}{2}\right) W_{i, Y} W_{i, X Y}-W_{i, X}\left(W_{i, X X}+\left(\frac{1-v}{2}\right) W_{i, Y Y}\right)+v \frac{\bar{h}_{i}}{h_{i}^{*}} W_{i, X} \\
\left(\frac{1+v}{2}\right) U_{i, X Y}+\left(\frac{1-v}{2}\right) V_{i, X X}+V_{i, Y Y}=-\left(\frac{1+v}{2}\right) W_{i, X} W_{i, X Y}-W_{i, Y}\left(\left(\frac{1-v}{2}\right) W_{i, X X}+W_{i, Y Y}\right)+\frac{\bar{h}_{l}}{h_{i}^{*}} W_{i, Y} \\
\frac{h_{i}^{* 3}}{12} W_{i, X X X X}+\frac{h_{i}^{* 3}}{6} W_{i, X X Y Y}+\frac{h_{i}^{* 3}}{12} W_{i, Y Y Y Y}=W_{i, X X}\left(h_{i}^{*} U_{i, X}+\frac{h_{i}^{*}}{2} W_{i, X}^{2}+v h_{i}^{*} V_{i, Y}-\bar{h}_{l} W_{i}+\frac{1}{2} v h_{i}^{*} W_{i, Y}^{2}\right)+(1- \\
v) h_{i}^{*} W_{i, X Y}\left(U_{i, Y}+V_{i, X}+W_{i, X} W_{i, Y}\right)+\left(h_{i}^{*} W_{i, Y Y}+\hat{h}_{i}\right)\left(v U_{i, X}+\frac{1}{2} v W_{i, X}^{2}+V_{i, Y}-L^{*} W_{i}+\frac{1}{2} W_{i, Y}^{2}\right)
\end{gathered}
$$

By discretizing the governing differential equations (49) using differential quadrature method and applying the Hadamard product and power, the formulation for these equations will be:

$$
\begin{align*}
& {[B X U][U]+\left(\frac{1+v}{2}\right)[A X V][A Y V][V]+\left(\frac{1-v}{2}\right)[B Y U][U]} \\
& =-\left(\frac{1+v}{2}\right)[A Y W][W] \circ[A X W][A Y W][W]-[A X W][W] \circ\left([B X W][W]+\left(\frac{1-v}{2}\right)[B Y W][W]\right) \\
& \quad+v \frac{\bar{h}_{i}}{h_{i}^{*}}[A X W][W] \\
& \left(\frac{1+v}{2}\right)[A X U][A Y U][U]+\left(\frac{1-v}{2}\right)[B X V][V]+[B Y V][V] \\
& \\
& =-\left(\frac{1+v}{2}\right)[A X W][W] \circ[A X W][A Y W][W]-[A Y W][W] \circ\left(\left(\frac{1-v}{2}\right)[B X W][W]+[B Y W][W]\right) \\
& \\
& \quad+\frac{\bar{h}_{i}}{h_{i}^{*}}[A Y W][W]
\end{align*}
$$

Where:
$[U],[V]$ and $[W]$ are square matrices for non-dimensional displacements $U, V$ and $W$ respectively.
$[A X U],[A X V],[A X W]$ are weighting coefficients of first order in $X$ - direction for $U, V$ and $W$ respectively.
$[B X U],[B X V],[B X W]$ are weighting coefficients of second order in $X-$ direction for $U, V$ and $W$ respectively.
[ $D X W$ ] are weighting coefficients of fourth order in $X$ - direction for $W$.
[AYU], [AYV], [AYW] are weighting coefficients of first order in $Y$ - direction for $U, V$ and $W$ respectively.
[BYU], $[B Y V],[B Y W]$ are weighting coefficients of second order in $Y$ - direction for $U, V$ and $W$ respectively.
[DYW] are weighting coefficients of fourth order in $Y$ - direction for $W$.
${ }^{\circ}$ denote Hadamard product.
$\forall^{\circ}$ denote Hadamard power.
For stacking the rows of each square matrix $[U],[V]$ and $[W]$ into one long vector the Kronecker product will be used, as following:

$$
\begin{gather*}
{\left[S_{1}\right][\bar{U}]+\left[S_{2}\right][\bar{V}]=-\left(\frac{1+v}{2}\right) h_{i}^{*}\left[S_{3}\right][\bar{W}] \circ\left[S_{4}\right][\bar{W}]-\left[S_{5}\right][\bar{W}] \circ\left(h_{i}^{*}\left[S_{6}\right][\bar{W}]+\left(\frac{1-v}{2}\right) h_{i}^{*}\left[S_{7}\right][\bar{W}]\right)+v \bar{h}_{l}\left[S_{5}\right][\bar{W}]} \\
{\left[S_{8}\right][\bar{U}]+\left[S_{9}\right][\bar{V}]=-\left(\frac{1+v}{2}\right) h_{i}^{*}\left[S_{5}\right][\bar{W}] \circ\left[S_{4}\right][\bar{W}]-\left[S_{3}\right][W] \circ\left(\left(\frac{1-v}{2}\right) h_{i}^{*}\left[S_{6}\right][W]+h_{i}^{*}\left[S_{7}\right][W]\right)+\bar{h}_{l}\left[S_{3}\right][W]} \\
{\left[S_{6}\right][\bar{W}] \circ\left(h_{i}^{*}\left[S_{11}\right][\bar{U}]+\frac{h_{i}^{*}}{2}\left[\left[S_{5}\right][\bar{W}]\right]^{\circ 2}+v h_{i}^{*}\left[S_{12}\right][\bar{V}]-\bar{h}_{l}[\bar{W}]+\frac{1}{2} v h_{i}^{*}\left[\left[S_{3}\right][\bar{W}]\right]^{\circ 2}\right)+(1-v) h_{i}^{*}\left[S_{4}\right][\bar{W}] \circ} \\
\left(\left[S_{13}\right][\bar{U}]+\left[S_{14}\right][\bar{V}]+\left[S_{5}\right][\bar{W}] \circ\left[S_{3}\right][\bar{W}]\right)+\left(h_{i}^{*}\left[S_{7}\right][\bar{W}] \circ+\widehat{h}_{i}\right)\left(v\left[S_{11}\right][\bar{U}]+\frac{1}{2} v\left[\left[S_{5}\right][\bar{W}]\right]^{\circ 2}+\left[S_{12}\right][\bar{V}]-L^{*}[\bar{W}]+\right. \\
\left.\frac{1}{2}\left[\left[S_{3}\right][\bar{W}]\right]^{\circ 2}\right)
\end{gather*}
$$

Where:
$\left[S_{1}\right]=h_{i}^{*}[B X U] \otimes[I]+\left(\frac{1-v}{2}\right) h_{i}^{*}[B Y U] \otimes[I]$
$\left[S_{2}\right]=\left(\frac{1+v}{2}\right) h_{i}^{*}[A X V] \otimes[A Y V]$
$\left[S_{3}\right]=[A Y W] \otimes[I]$
$\left[S_{4}\right]=[A X W] \otimes[A Y W]$
$\left[S_{5}\right]=[A X W] \otimes[I]$
$\left[S_{6}\right]=[B X W] \otimes[I]$
$\left[S_{7}\right]=[B Y W] \otimes[I]$
$\left[S_{8}\right]=\left(\frac{1+v}{2}\right) h_{i}^{*}[A X U] \otimes[A Y U]$

$$
\begin{aligned}
& {\left[S_{9}\right]=\left(\frac{1-v}{2}\right) h_{i}^{*}[B X V] \otimes[I]+h_{i}^{*}[B Y V] \otimes[I]} \\
& {\left[S_{10}\right]=\frac{h_{i}^{* 3}}{12}[D X W] \otimes[I]+\frac{h_{i}^{* 3}}{6}[B X W] \otimes[B Y W]+\frac{h_{i}^{* 3}}{12}[D Y W] \otimes[I]} \\
& {\left[S_{11}\right]=[A X U] \otimes[I]} \\
& {\left[S_{12}\right]=[A Y V] \otimes[I]} \\
& {\left[S_{13}\right]=[A Y U \otimes[I]} \\
& {\left[S_{14}\right]=[A X V] \otimes[I]}
\end{aligned}
$$

## 6. Verification of the Proposed Solution

To verify the solution derived herein by differential quadrature method, consider a simply supported cylindrical shell without delamination (i.e $\beta=0$ ) of radius $R=0.6 \mathrm{~m}$, length $L=1 \mathrm{~m}$ and Poisson's ratio $v=0.3$. One should note that to obtain the critical buckling load the transverse displacement $w$ should be put in the common mode of the buckling shape to enable one to treat the obtained governing
equation as an Eigen value problem, hence:

$$
w=C \cos (m y) \sin \left(\frac{n \pi R}{L} x\right)
$$

Where: $m, n$ are integer constants
The obtained critical buckling loads at different thicknesses were compared with those of the exact solution [1]. Table 1 shows the percentage of the error between the exact solution and the differential quadrature solution.

Table 1. Verification of the Proposed Solution.

| Non dimensional thickness $\left(\boldsymbol{h}_{\boldsymbol{i}}^{*}\right)$ | Non dimensional critical buckling load $\left(\boldsymbol{N}_{\boldsymbol{C r} \boldsymbol{r}}^{*}\right)$ | Differential quadrature solution | \% Error |
| :--- | :--- | :--- | :--- |
|  | Exact solution | 0.001975 | 1.25 |
| 0.002 | 0.00200 | 0.004691 | 2.27 |
| 0.005 | 0.00480 | 0.006148 |  |

## 7. Results and Discussion

To study the delamination growth, the effect of the delamination length and depth on the critical buckling load and energy release rate is studied as shown in the following figures.


Figure 4. Effect of Delamination Length on Critical Buckling Load.


Figure 5. Effect of Delamination Length on Energy Release Rate.


Figure 6. Effect of Delamination Thickness on Energy Release Rate.

From the above Figure 4, one can note that the critical buckling load decreases as the delamination length increases, since the critical buckling load decreases by $51.77 \%$ at ( $\alpha=$ $0.5)$ and by $82.69 \%$ at ( $\alpha=0.2$ ) as the delamination length becomes half the cylindrical shell length. It can be noted that the critical buckling load decreases as the delamination thickness parameter $(\alpha)$ decreases, since the critical buckling load decreases by $30.92 \%$ as the delamination thickness parameter decreases from 0.5 to 0.2 .

From previous Figure 5, one can note that the energy rate increases with delamination length. This shows that the delamination growth is easier to occur for longer delamination, since at $\alpha=0.2$ the axial load required to release energy for $\beta=0.3$ is $50 \%$ of the load required to release energy for $\beta=0.1$.

From the above Figure 6, one can note that the energy
release rate increases as the delamination thickness parameter decreases. This shows that delamination growth is easier to occur for shallow delamination, since at $\beta=0.3$ the axial load required to release energy for $\alpha=0.2$ is $44.444 \%$ of the load required to release energy for $\alpha=0.5$.

## 8. Conclusion

The Generalized Differential Quadrature method is seen to yield excellent results for the case treated, even when only a few grid points are used for the evaluation. Also, a simple way of the treatment of clamped - free boundary condition is applied. The critical buckling load of cylindrical shells decreases as the delamination thickness parameter ( $\alpha$ ) decreases. The delamination growth is easier to occur for longer and shallow delamination.

## Notations

| $A_{i j}^{(i)}$ | Extension rigidity |
| :---: | :---: |
| $B_{i j}^{(i)}$ | Coupling rigidity |
| $C_{i j}{ }^{(m)}$ | Weighting coefficient |
| $D_{i j}^{(i)}$ | Bending rigidity |
| E | Modulus of elasticity |
| $\mathrm{G}_{\mathrm{i}}$ | Energy release rate |
| h | Overall thickness |
| $\mathrm{h}_{\mathrm{i}}$ | Thickness of delaminated regions |
| L | Length of the cylindrical shell |
| $M_{i x}, M_{i y}, M_{i x y}$ | Couples stress |
| $N_{i x}, N_{i y}, N_{i x y}$ | Membrane stress resultant |
| $\bar{N}$ | Uniform axial load |
| $N^{*}$ | Non - dimensional axial load |
| R | Radius of the mid surface of circular cylindrical shell |
| $\mathrm{U}_{\mathrm{i}}$ | Strain energy |
| $u_{i}, v_{i}, w_{i}$ | Axial, circumference and radial displacements of the mid surface of the cylindrical shell |
| $u_{i}, \nu_{i}, w_{i}$ | Axial, circumference and radial displacements of any point within thickness of the cylindrical shell |
| $\mathrm{U}_{\mathrm{i}}, \mathrm{V}_{\mathrm{i}}, \mathrm{W}_{\mathrm{i}}$ | Non-dimensional axial, circumference and radial displacements of the mid surface of the cylindrical shell |
| $x, y, z$ | Coordinate system |
| $X, Y, Z$ | Non-dimensional coordinate system |
| $\mathrm{Z}^{*}$ | Distance measured from the shell mid-surface to the delamination interface |
| $\beta$ | Delamination length parameter |
| $\pi$ | Total potential energy |
| $v$ | Poisson's ratio |
| $\rho$ | Mass density of shell material |
| $\ell$ | Delamination position measured from the left of the shell |
| $\mathcal{E}_{\text {ix }}, \mathcal{E}_{\text {ix }}, \mathcal{E}_{\text {ixy }}$ | Strains of the mid surface of the cylindrical shell |
| $\overline{\mathcal{E}}_{\text {ix }}, \bar{\varepsilon}_{\text {iy }}, \bar{\varepsilon}_{\text {ixy }}$ | Strains of any point within thickness of the cylindrical shell |
| $\kappa_{i x}, \kappa_{i y}, \kappa_{i x y}$ | Curvature of the mid surface of the cylindrical shell |

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