# The Effect of Oblateness up to Zonal Harmonic $\mathbf{J}_{4}$ on <br> the Positions and Linear Stability of the Collinear Libration Points in the Photogravitational ER3BP 

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#### Abstract

We have investigated the motion of an infinitesimal body in the elliptic restricted three-body problem (ER3BP) when both primaries are sources of radiation as well as oblate spheroids with oblateness up to zonal harmonic $\mathrm{J}_{4}$. We highlight the effects of the said parameters on the locations of the collinear equilibrium points of 61 CYGNI and STRUVE 2398. It is also found that under the combined effect of the zonal harmonics $\left(\mathrm{J}_{2} \& \mathrm{~J}_{4}\right)$, the collinear points $\mathrm{L}_{1}$ and $\mathrm{L}_{2}$ move away from the bigger primary with the increase in oblateness, while $\mathrm{L}_{3}$ moves closer to the primaries. It is also seen that in the case of the binary system 61 Cygni, the effect of the zonal harmonics $\left(\mathrm{J}_{2} \& \mathrm{~J}_{4}\right)$ on the positions of $\mathrm{L}_{1}$ and $\mathrm{L}_{3}$ is not observable when compared with the binary system Struve 2398. It is further observed that the oblateness does not change the nature of stability of collinear points and they remain unstable.


## Keywords

Celestial Mechanics, ER3BP, Oblateness, Collinear Points

## 1. Introduction

The restricted three-body problem (R3BP) with the assumptions of the sphericity of the participating bodies and circularity of their orbits has continued to fascinate and intrigue researchers. Two spherical massive bodies (primaries) move in circular orbits under their mutual gravitational attraction influencing but not being influenced by the third massless body. In such a system, five co-planar equilibrium points exist; three collinear with the line joining the primaries and two form equilateral triangles with the primaries. The collinear points have been shown to be generally unstable, while the triangular points are conditionally stable [1-5]

When the orbits of the primaries are elliptic, called the elliptic restricted three-body problem (ER3BP), a nonuniformly rotating-pulsating coordinate system is commonly used. This new coordinate system has the felicitous property
that, the positions of the primaries are fixed; however, the Hamiltonian is explicitly time-dependent [6]. Such an oscillating coordinate system has been introduced by using the variable distance between the primaries as a unit of length of the system by which distances are divided. [7] studied the effect of a small perturbation in the coriolis force on the stability of the triangular points taking the centrifugal force as constant. He concluded that the collinear points remain unstable and the coriolis force is a stabilizing force when considering the stability of triangular points. He further obtained a relation between the value of the critical mass parameter $\mu_{c}$ and the change $\varepsilon$ in the coriolis force as; $\mu_{c}=\mu_{0}+\frac{16 \varepsilon}{\sqrt[3]{69}}$

The work of [7] was extended by [1] to include the effect of perturbations $\varepsilon$ and $\varepsilon^{\prime}$ in the coriolis and centrifugal forces respectively and found that the collinear points remain unstable. They also obtained a relation for triangular points changes $\varepsilon, \varepsilon^{\prime}$ and $\mu_{c}$ as: $\mu_{c}=\mu_{0}+\frac{4\left(36 \varepsilon-19 \varepsilon^{\prime}\right)}{27 \sqrt{69}}$

In his study of linear stability of triangular equilibrium points of the photogravitational restricted three-body problem when the more massive primary (Sun) is a source of radiation and the smaller one is an oblate spheroid, [8] showed that the triangular points have long or short-periodic retrograde elliptical orbits and the value of the critical mass parameter decreases with an increase in oblateness and radiation force. [9] considered the bigger primary as an oblate spheroid and the smaller primary as a triaxial rigid body in the R3BP; they confirmed that the triangular points have long or short periodic elliptical orbits in the same range of the mass parameter and the nature of collinear and triangular points remain unstable and stable respectively for some values of mass parameters. [10] considered both primaries as triaxial rigid bodies as well as sources of radiation in their study of the existence and stability of libration points. They maintained that the three collinear points are unstable and the two triangular points are stable for certain values of mass parameter.

A number of studies have been carried out on the collinear liberation points of the circular restricted three-body problem (CR3BP). [11] examined the existence and linear stability of liberation points in the axisymmetric restricted three-body problem with radiating primaries and found that, the collinear points remain unstable. Collinear points exhibiting characters of stability were studied by [2] and [12]. They admit that, the inner collinear point can be stable under certain conditions. In the same analogy, [13] stated that, the collinear points can be stable in the Lyapunov sense in the case of a fourth order resonance in the presence of the radiation of both primaries. A number of researchers have affirmed that the collinear points are usually unstable [14-18]. [19] examined the equilibrium points and their stability in the R3BP with oblateness and variable masses. They confirmed that, the collinear points are stable due to k (kappa). [14] investigated the effects of the luminosity and oblateness of both primary bodies on the collinear libration points of the binary systems Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB and Xi Bootis moving in elliptic orbits around their common centre of mass. Also, [15] investigated the effect of the triaxiality of the bigger primary on the collinear libration points of binary pulsars. They found that, the positions of collinear points are affected by the eccentricity, oblateness, radiation and triaxiality factors: These points however remain unstable.

Our aim is to study the effect of oblateness up to zonal harmonic $\mathrm{J}_{4}$ on the positions and linear stability of the collinear libration points. We have chosen two binary system (61Cygni and Struve 2398) for our numerical computations.

This paper is organised in six sections; section 1 is the introduction; section 2 deals with the equations of motion; section 3 focuses on the positions of collinear points. The stability of collinear points is examined in section 4 ; while section 5 deals with the numerical applications, section 6 focuses on discussion and conclusion.

## 2. Equations of Motion

The equations of motion of an infinitesimal mass, in the ER3BP with oblate as well as luminous primaries, can be written in the dimensionless-pulsating coordinate system $(\xi, \eta, \zeta)$ following [20] as;

$$
\begin{gather*}
\xi^{\prime \prime}-2 \eta^{\prime}=\Omega_{\xi}, \eta^{\prime \prime}+2 \xi^{\prime}=\Omega_{\eta}, \zeta^{\prime \prime}=\Omega_{\zeta}  \tag{1}\\
\Omega=\frac{1}{\left(1-\mathrm{e}^{2}\right)^{1 / 2}}\left[\frac{1}{2}\left(\xi^{2}+\eta^{2}\right)+\frac{1}{\mathrm{n}^{2}}\left\{\frac{(1-\mu) q_{1}}{r_{1}}+\frac{(1-\mu) A_{1} q_{1}}{2 r_{1}^{3}}-\right.\right. \\
\frac{\frac{3}{8}}{\left.\left.\frac{(1-\mu) A_{2} q_{1}}{r_{1}^{5}}+\frac{\mu q_{2}}{r_{2}}+\frac{\mu B_{1} q_{2}}{2 r_{2}^{3}}-\frac{3 \mu B_{2} q_{2}}{8 r_{2}^{5}}\right\}\right]} \tag{2}
\end{gather*}
$$

The mean motion, $n$, is given by

$$
\begin{gather*}
\mathrm{n}^{2}=\frac{\left(1+e^{2}\right)^{1 / 2}}{a\left(1-e^{2}\right)}\left[1+\frac{3}{2} A_{1}+\frac{3}{2} B_{1}-\frac{15}{8} A_{2}-\frac{15}{8} B_{2}\right]=\frac{1}{a}(1+ \\
\left.\frac{3 e^{2}}{2}+\frac{3 A_{1}}{2}+\frac{3 B_{1}}{2}-\frac{15 A_{2}}{8}-\frac{15 B_{2}}{8}\right) \tag{3}
\end{gather*}
$$

and

$$
\begin{gather*}
r_{1}^{2}=(\xi+\mu)^{2}+\eta^{2}+\zeta^{2}, r_{2}^{2}=(\xi+\mu-1)^{2}+\eta^{2}+\zeta^{2} \\
\xi_{1}=-\mu, \xi_{2}=1-\mu  \tag{4}\\
0<\mu=\frac{m_{2}}{m_{1}+m_{2}}<\frac{1}{2} \tag{5}
\end{gather*}
$$

Here, $m_{1}, m_{2}$ are the masses of the bigger and smaller primaries positioned at the points $\left(\xi_{i}, 0,0\right), i=1,2 ;, \mathrm{q}_{1}, \mathrm{q}_{2}$ are their radiation factors; $r_{i}$, are their distances from the infinitesimal mass; respectively; a and e are the semi-major axis and eccentricity of the orbits respectively; $A_{i}=$ $\mathrm{J}_{2 \mathrm{i}} \mathrm{R}_{1}{ }^{2}, \mathrm{~B}_{\mathrm{i}}=\overline{\mathrm{J}}_{2 \mathrm{i}} \mathrm{R}_{2}{ }^{2} \mathrm{~A}_{\mathrm{i}}, \mathrm{B}_{\mathrm{i}} \ll 1(\mathrm{i}=1,2)$ characterize the zonal harmonic oblateness of the bigger and smaller primaries whose mean radii are $R_{1}$ and $R_{2}$ respectively.

## 3. Location of Collinear Points

The equilibrium points are the solutions of the equations $\Omega_{\xi}=\Omega_{\eta}=\Omega_{\zeta}=0$, which yield

$$
\begin{gather*}
\xi-\frac{1}{\mathrm{n}^{2}}\left\{\frac{(1-\mu)\left(\xi-\xi_{1}\right)}{r_{1}^{3}} q_{1}+\frac{3(1-\mu)\left(\xi-\xi_{1}\right)}{2 r_{1}^{5}} A_{1} q_{1}-\frac{15(1-\mu)\left(\xi-\xi_{1}\right)}{8 r_{1}^{7}} A_{2} q_{1}+\frac{\mu\left(\xi-\xi_{2}\right) q_{2}}{r_{2}^{3}}+\frac{3 \mu\left(\xi-\xi_{2}\right) B_{1} q_{2}}{2 r_{2}^{5}}-\frac{15 \mu\left(\xi-\xi_{2}\right)}{8 r_{2}^{7}} B_{2} q_{2}\right\}=0 \\
\eta\left[1-\frac{1}{\mathrm{n}^{2}}\left\{\frac{(1-\mu)}{r_{1}{ }^{3}} q_{1}+\frac{3(1-\mu)}{2 r_{1}{ }^{5}} A_{1} q_{1}-\frac{15(1-\mu)}{8 r_{1}{ }^{7}} A_{2} q_{1}+\frac{\mu}{r_{2}{ }^{3}} q_{2}+\frac{3 \mu}{2 r_{2}{ }^{5}} B_{1} q_{2}-\frac{15 \mu}{8 r_{2}{ }^{7}} B_{2} q_{2}\right\}\right]=0 \\
\zeta\left[\left\{\frac{(1-\mu)}{r_{1}{ }^{3}} q_{1}+\frac{3(1-\mu)}{2 r_{1}{ }^{5}} A_{1} q_{1}-\frac{15(1-\mu)}{8 r_{1}{ }^{7}} A_{2} q_{1}+\frac{\mu q_{2}}{r_{2}{ }^{3}}+\frac{3 \mu B_{1} q_{2}}{2 r_{2}^{5}}-\frac{15 \mu}{8 r_{2}^{7}} B_{2} q_{2}\right\}\right]=0 \tag{6}
\end{gather*}
$$

From equation (4) with $\eta=\zeta=0$ and the first of equations (6), we have;

$$
\begin{equation*}
n^{2} \xi-\left[\frac{(1-\mu)(\xi+\mu) q_{1}}{|\xi+\mu|^{3}}+\frac{3(1-\mu)(\xi+\mu) q_{1} A_{1}}{2|\xi+\mu|^{5}}-\frac{15(1-\mu)(\xi+\mu) q_{1} A_{2}}{8|\xi+\mu|^{7}}+\frac{\mu(\xi+\mu-1) q_{2}}{|\xi+\mu-1|^{3}}+\frac{3 \mu(\xi+\mu-1) q_{2} B_{1}}{2|\xi+\mu-1|^{5}}-\frac{15 \mu(\xi+\mu-1) q_{2} B_{2}}{8|\xi+\mu-1|^{7}}\right]=0 \tag{7}
\end{equation*}
$$

To obtain locations of collinear points on the $\xi$ - axis, we divide the orbital plane into three parts; $\xi>\xi_{2}, \xi_{1}<\xi<\xi_{2}$ and $\xi_{1}>\xi$ with respect to their primaries, given that $\eta=\zeta=0$. These three points are considered in Cases I, II and III respectively.

Case I: Let the collinear point $L_{1}$ be on the RHS of the smaller primary at a distance $\rho$ from it on the $\xi-$ axis (i.e $\xi>\xi_{2}$ ).

| $m_{1}=1-\mu$ | $m_{2}=u$ | $L_{1}$ |
| :--- | :--- | :--- |
| $\left(\xi_{1}, 0\right)$ | 0 | $\left.\left(\xi_{2}, 0\right) \longleftrightarrow, 0\right)$ |

Let us consider $\xi-\xi_{2}=\rho$
But $\xi_{2}-\xi_{1}=1$, then $\xi-1-\xi_{1}=\rho$. Therefore $\xi-\xi_{1}=1+\rho$
Since $O$ is the centre of mass
$(1-\mu) \xi_{1}+\mu \xi_{2}=0 ; \xi_{1}+\mu\left(\xi_{2}-\xi_{1}\right)=0\left(\right.$ but $\left.\xi_{2}-\xi_{1}=1\right)$
Then; $\xi_{1}+\mu=0, \xi_{1}=-\mu$

$$
\begin{equation*}
\text { Therefore } ; \xi=\xi_{1}+1+\rho=1+\rho-\mu \tag{8}
\end{equation*}
$$

Substituting equation (8) into equation (7), yields,

$$
\begin{gather*}
8 n^{2}(1+\rho-\mu)(1+\rho)^{6} \rho^{6}-8(1-\mu)(1+\rho)^{4} \rho^{6} q_{1}-12(1-\mu)(1+\rho)^{2} \rho^{6} q_{1} A_{1}+15(1-\mu) \rho^{6} q_{1} A_{2}-8 \mu(1+ \\
\rho)^{6} \rho^{4} q_{2}-12 \mu(1+\rho)^{6} \rho^{2} q_{2} B_{1}+15 \mu(1+\rho)^{6} q_{2} B_{2}=0 \tag{9}
\end{gather*}
$$

Case II: Let the collinear point $L_{2}$ be on the LHS of the smaller primary at a distance $\rho$ from it on the $\xi-$ axis (i.e $\xi_{1}<\xi<$ $\xi_{2}$ ).


Where $\xi_{1}=-\mu, \xi_{2}=1-\mu$ then; $\xi=\xi_{2}-\rho=1-\mu-\rho$;

$$
\begin{equation*}
r_{2}=\rho, r_{1}=1-\rho,\left(\xi-\xi_{1}\right)=1-\rho,\left(\xi-\xi_{2}\right)=-\rho \tag{10}
\end{equation*}
$$

Substituting equation (10) in equation (7), yields,

$$
\begin{equation*}
n^{2}(1-\mu-\rho)-\left[\frac{(1-\mu) q_{1}}{(1-\rho)^{2}}+\frac{3(1-\mu) q_{1} A_{1}}{2(1-\rho)^{4}}-\frac{15(1-\mu) q_{1} A_{2}}{8(1-\rho)^{6}}-\frac{\mu q_{2}}{\rho^{2}}-\frac{3 \mu q_{2} B_{1}}{2 \rho^{4}}+\frac{15 \mu q_{2} B_{2}}{8 \rho^{6}}\right]=0 \tag{11}
\end{equation*}
$$

Multiplying by $8(1-\rho)^{6} \rho^{6}$ we obtain,

$$
\begin{gather*}
8 n^{2}(1-\mu-\rho)(1-\rho)^{6} \rho^{6}-8(1-\mu)(1-\rho)^{4} \rho^{6} q_{1}-12(1-\mu)(1-\rho)^{2} \rho^{6} q_{1} A_{1}+15(1-\mu) \rho^{6} q_{1} A_{2}+8 \mu(1- \\
\rho)^{6} \rho^{4} q_{2}+12 \mu(1-\rho)^{6} \rho^{2} q_{2} B_{1}-15 \mu(1-\rho)^{6} q_{2} B_{2}=0 \tag{12}
\end{gather*}
$$

Case III: Let the collinear point $L_{3}$ be on the LHS of the bigger primary at a distance $\rho$ from it on the $\xi-$ axis (i.e $\xi_{1}>\xi$ ).

where $\xi_{1}=-\mu, \xi_{2}=1-\mu$ then

$$
\begin{gather*}
\xi=\xi_{1}-\rho=-\mu-\rho \\
r_{1}=\rho, r_{2}=r_{1}+1=1+\rho  \tag{13}\\
\left(\xi-\xi_{1}\right)=-\rho,\left(\xi-\xi_{2}\right)=-(1+\rho)
\end{gather*}
$$

Substituting (13) into equation (7), we have

$$
\begin{equation*}
n^{2}(-\mu-\rho)-\left[\frac{(1-\mu)(-\rho) q_{1}}{\rho^{3}}+\frac{3(1-\mu)(-\rho) q_{1} A_{1}}{2 \rho^{5}}-\frac{15(1-\mu)(-\rho) q_{1} A_{2}}{8 \rho^{7}}+\frac{\mu(-(1+\rho)) q_{2}}{(1+\rho)^{3}}+\frac{3 \mu(-(1+\rho)) q_{2} B_{1}}{2(1+\rho)^{5}}-\frac{15 \mu(-(1+\rho)) q_{2} B_{2}}{8(1+\rho)^{7}}\right]=0 \tag{14}
\end{equation*}
$$

Multiplying by $8(1+\rho)^{6}$, we have

$$
\begin{gather*}
8 n^{2}(-\mu-\rho)(1+\rho)^{6} \rho^{6}+8(1-\mu)(1+\rho)^{6} \rho^{4} q_{1}+12(1-\mu)(1+\rho)^{6} \rho^{2} q_{1} A_{1}-15(1-\mu)(1+\rho)^{6} q_{1} A_{2}+8 \mu(1+ \\
\rho)^{4} \rho^{6} q_{2}+12 \mu(1+\rho)^{2} \rho^{6} q_{2} B_{1}-15 \mu \rho^{6} q_{2} B_{2}=0 \tag{15}
\end{gather*}
$$

The real root of each of equations (9), (12) and (15) gives the position of collinear point.

## 4. Stability of Collinear Points

To examine the stability of the collinear points, we consider the characteristic equation of the system given below by [20];

$$
\begin{equation*}
\lambda^{4}-\left(\Omega_{\xi \xi}^{0}+\Omega_{\eta \eta}^{0}-4\right) \lambda^{2}+\Omega_{\xi \xi}^{0} \Omega_{\eta \eta}^{0}-\left(\Omega_{\xi \eta}^{0}\right)^{2}=0 \tag{16}
\end{equation*}
$$

The second derivatives are thus;

$$
\begin{aligned}
\Omega_{\xi \xi}=\frac{1}{\left(1-\mathrm{e}^{2}\right)^{1 / 2}} & {\left[1-\frac{1}{\mathrm{n}^{2}}\left\{\frac{(1-\mu) q_{1}}{r_{1}^{3}}+\frac{3(1-\mu)}{2 r_{1}^{5}} A_{1} q_{1}-\frac{15}{8 r_{1}^{7}}(1-\mu) A_{2} q_{1}+\frac{\mu q_{2}}{r_{2}^{3}}+\frac{3 \mu B_{1} q_{2}}{2 r_{2}^{5}}-\frac{15}{8 r_{2}^{7}} \mu B_{2} q_{2}\right\}\right.} \\
& +\frac{1}{\mathrm{n}^{2}}\left\{\frac{3(1-\mu)\left(\xi-\xi_{1}\right)^{2} q_{1}}{r_{1}{ }^{5}}+\frac{15(1-\mu)\left(\xi-\xi_{1}\right)^{2}}{2 r_{1}{ }^{7}} A_{1} q_{1}-\frac{105(1-\mu)\left(\xi-\xi_{1}\right)^{2} A_{2} q_{1}}{8 r_{1}{ }^{9}}+\frac{3 \mu\left(\xi-\xi_{2}\right)^{2} q_{2}}{r_{2}{ }^{5}}\right. \\
& \left.\left.+\frac{15 \mu B_{1}\left(\xi-\xi_{2}\right)^{2} q_{2}}{2 r_{2}{ }^{7}}-\frac{105 \mu B_{2}\left(\xi-\xi_{2}\right)^{2} q_{2}}{8 r_{2}{ }^{9}}\right\}\right]
\end{aligned}
$$

$$
\begin{align*}
& \Omega_{\eta \eta}= \frac{1}{\left(1-\mathrm{e}^{2}\right)^{1 / 2}}\left[1-\frac{1}{\mathrm{n}^{2}}\left\{\frac{(1-\mu) q_{1}}{r_{1}^{3}}+\frac{3(1-\mu)}{2 r_{1}^{5}} A_{1} q_{1}-\frac{15}{8 r_{1}^{7}}(1-\mu) A_{2} q_{1}+\frac{\mu q_{2}}{r_{2}^{3}}+\frac{3 \mu B_{1} q_{2}}{2 r_{2}^{5}}-\frac{15}{8 r_{2}^{7}} \mu B_{2} q_{2}\right\}\right. \\
&\left.+\frac{1}{\mathrm{n}^{2}}\left\{\frac{3(1-\mu) q_{1}}{r_{1}{ }^{5}}+\frac{15(1-\mu)}{2 r_{1}{ }^{7}} A_{1} q_{1}-\frac{105(1-\mu) A_{2} q_{1}}{8 r_{1}{ }^{9}}+\frac{3 \mu q_{2}}{r_{2}{ }^{5}}+\frac{15 \mu B_{1} q_{2}}{2 r_{2}{ }^{7}}-\frac{105 \mu B_{2} q_{2}}{8 r_{2}{ }^{9}}\right\} \eta^{2}\right] \\
& \Omega_{\xi \eta}=\frac{1}{\left(1-\mathrm{e}^{2}\right)^{1 / 2}} \cdot \frac{1}{\mathrm{n}^{2}}\left\{\begin{array}{c}
\frac{3(1-\mu)\left(\xi-\xi_{1}\right) q_{1}}{r_{1}{ }^{5}}+\frac{15(1-\mu)\left(\xi-\xi_{1}\right)}{2 r_{1}{ }^{7}} A_{1} q_{1}-\frac{105(1-\mu)\left(\xi-\xi_{1}\right) A_{2} q_{1}}{8 r_{1}{ }^{9}}+ \\
\frac{3 \mu\left(\xi-\xi_{2}\right) q_{2}}{r_{2}{ }^{5}}+\frac{15 \mu B_{1}\left(\xi-\xi_{2}\right) q_{2}}{2 r_{2}{ }^{7}}-\frac{105 \mu B_{2}\left(\xi-\xi_{2}\right) q_{2}}{8 r_{2}{ }^{9}}
\end{array}\right\} \eta \tag{17}
\end{align*}
$$

Now, for the stability of a collinear point exist on the $\xi$-axis, i.e. $\eta=\zeta=0$, then,

$$
\begin{equation*}
r_{1}=|\xi+\mu|, r_{2}=|\xi+\mu-1| \tag{18}
\end{equation*}
$$

Substituting equation (18) In the first of equation (17)
$\Omega_{\xi \xi}^{0}=\left(1-e^{2}\right)^{-1 / 2}\left[1+\frac{2}{n^{2}}\left(\frac{(1-\mu) q_{1}}{|\xi+\mu|^{3}}+\frac{3(1-\mu) q_{1} A_{1}}{|\xi+\mu|^{5}}-\frac{45(1-\mu) q_{1} A_{2}}{8|\xi+\mu|^{7}}+\frac{\mu q_{2}}{|\xi+\mu-1|^{3}}+\frac{3 \mu q_{2} B_{1}}{|\xi+\mu-1|^{5}}-\frac{45 \mu q_{2} B_{2}}{8|\xi+\mu-1|^{7}}\right)\right]>0$
For the second and third equations of (17) with $\eta=0$ we have

$$
\begin{gathered}
\Omega_{\eta \eta}^{0}=\left(1-e^{2}\right)^{-1 / 2}\left[1-\frac{1}{n^{2}}\left(\left(n^{2} \xi-\frac{3(1-\mu) q_{1} A_{1}}{2 r_{1}{ }^{4}}+\frac{15(1-\mu) q_{1} A_{2}}{8 r_{1}{ }^{6}}-\frac{\mu q_{2}}{r_{2}{ }^{2}}-\frac{3 \mu q_{2} B_{1}}{2 r_{2}{ }^{4}}+\frac{15 \mu q_{2} B_{2}}{8 r_{2}{ }^{6}}\right) \frac{1}{r_{1}}+\frac{3(1-\mu) q_{1} A_{1}}{2 r_{1}{ }^{5}}-\frac{15(1-\mu) q_{1} A_{2}}{8 r_{1}{ }^{7}}+\right.\right. \\
\left.\left.\frac{\mu q_{2}}{r_{2}{ }^{3}}+\frac{3 \mu q_{2} B_{1}}{2 r_{2}{ }^{5}}-\frac{15 \mu q_{2} B_{2}}{8 r_{2}{ }^{7}}\right)\right]
\end{gathered}
$$

And $\Omega_{\xi \eta}^{0}=0$.
The first of equation (6) with $\eta=0$ can be written as

$$
\begin{equation*}
\frac{(1-\mu) q_{1}}{r_{1}{ }^{2}}=n^{2} \xi-\frac{3(1-\mu) q_{1} A_{1}}{2 r_{1}{ }^{4}}+\frac{15(1-\mu) q_{1} A_{2}}{8 r_{1}{ }^{6}}-\frac{\mu q_{2}}{r_{2}{ }^{2}}-\frac{3 \mu q_{2} B_{1}}{2 r_{2}{ }^{4}}+\frac{15 \mu q_{2} B_{2}}{8 r_{2}{ }^{6}} \tag{19}
\end{equation*}
$$

Using equation (19)

$$
\Omega_{\eta \eta}^{0}=\left(1-e^{2}\right)^{-1 / 2}\left[1-\frac{1}{n^{2}}\left(n^{2} \xi-\frac{3(1-\mu) q_{1} A_{1}}{2 r_{1}{ }^{4}}+\frac{15(1-\mu) q_{1} A_{2}}{8 r_{1}{ }^{6}}-\frac{\mu q_{2}}{r_{2}{ }^{2}}-\frac{3 \mu q_{2} B_{1}}{2 r_{2}{ }^{4}}+\frac{15 \mu q_{2} B_{2}}{8 r_{2}{ }^{6}}\right) \frac{1}{r_{1}}-\frac{1}{n^{2}}\left(\frac{3(1-\mu) q_{1} A_{1}}{2 r_{1}{ }^{5}}-\frac{15(1-\mu) q_{1} A_{2}}{8 r_{1}{ }^{7}}+\right.\right.
$$

$$
\begin{equation*}
\left.\left.\frac{\mu q_{2}}{r_{2}{ }^{3}}+\frac{3 \mu q_{2} B_{1}}{2 r_{2}{ }^{5}}-\frac{15 \mu q_{2} B_{2}}{8 r_{2}{ }^{7}}\right)\right] \tag{20}
\end{equation*}
$$

Neglecting higher order terms of a, $e^{2}, A_{1}, B_{1}, A_{2}$ and $B_{2}$ we have,

$$
\begin{aligned}
\Omega_{\eta \eta}^{0}=\left[\frac{\mu}{r_{1}}+\frac{\mu e^{2}}{2 r_{1}}\right. & +\frac{a \mu q_{2}}{r_{1} \cdot r_{2}^{2}}\left(1-\frac{3 A_{1}}{2}+\frac{15 A_{2}}{8}-e^{2}+\frac{3 B_{1}}{2}\left(\frac{1}{r_{2}^{2}}-1\right)+\frac{15 B_{2}}{8}\left(1-\frac{1}{r_{2}^{3}}\right)\right) \\
& \left.-\frac{a \mu q_{2}}{r_{2}^{3}}\left(1-\frac{3 A_{1}}{2}+\frac{15 A_{2}}{8}-e^{2}-\frac{3 B_{1}}{2}\left(1-\frac{1}{r_{2}^{2}}\right)-\frac{15 B_{2}}{8}\left(\frac{1}{r_{2}^{4}}-1\right)\right)\right]
\end{aligned}
$$

Since $\mu<\frac{1}{2}, A_{i}, B_{i}, e^{2} \ll 1, r_{1}>1, r_{2}<1$ where $i=1,2$. we have

$$
\begin{gather*}
\Omega_{\eta \eta}^{0}=\left[\frac{\mu}{r_{1}}+\frac{\mu e^{2}}{2 r_{1}}+\frac{a \mu q_{2}}{r_{1} \cdot r_{2}^{2}}\left(1-\frac{3 A_{1}}{2}+\frac{15 A_{2}}{8}-e^{2}+\frac{3 B_{1}}{2}\left(\frac{1}{r_{2}^{2}}-1\right)+\frac{15 B_{2}}{8}\left(1-\frac{1}{r_{2}^{3}}\right)\right)-\frac{a \mu q_{2}}{r_{2}^{3}}\left(1-\frac{3 A_{1}}{2}+\frac{15 A_{2}}{8}-e^{2}-\frac{3 B_{1}}{2}\left(1-\frac{1}{r_{2}^{2}}\right)-\right.\right. \\
\left.\left.\frac{15 B_{2}}{8}\left(\frac{1}{r_{2}^{4}}-1\right)\right)\right]<0 \\
\Omega_{\xi \eta}^{0}=0 \text { since } \eta=0 . \tag{22}
\end{gather*}
$$

Therefore, for the collinear points lying in the interval $\left(\xi>\xi_{2}\right),\left(\xi_{1}<\xi<\xi_{2}\right)$ and $\left(\xi_{1}>\xi\right)$ respectively with respect to their primaries, given that $\eta=\zeta=0$, we have $\Omega_{\xi \xi}^{0}>0, \Omega_{\eta \eta}^{0}<0$ and $\Omega_{\xi \eta}^{0}=0$.

Since $\Omega_{\xi \xi}^{0} \Omega_{\eta \eta}^{0}-\left(\Omega_{\xi \eta}^{0}\right)^{2}<0$, its discriminant is positive and the roots can be expressed as $\lambda_{1,2}= \pm a$ and $\lambda_{3,4}= \pm i b$ where $a$ and $b$ are real. This confirms that, the motion in the neighborhood of the collinear points is unstable since it is not

## 5. Numerical Applications

The collinear points denoted by $L_{1}, L_{2}, L_{3}$ are evidenced by cases I, II and II respectively. Using Equations (9), (12) and (15), for various oblateness up to $\mathrm{J}_{4}$, mass ratio $(\mu)$, mean motion ( $n$ ) radiation pressure ( $\mathrm{q}_{1}$ and $\mathrm{q}_{2}$ ), we compute numerically using MATHEMATICA software, the positions of the collinear points as given in table 2.

Table 1. Numerical data for the binary systems.

| Binary system | $\mathbf{M}_{1}\left(\mathbf{M}_{\text {sun }}\right)$ | $\mathbf{M}_{\mathbf{2}}\left(\mathbf{M}_{\text {sun }}\right)$ | $\mathbf{L}_{1}\left(\mathbf{L}_{\text {sun }}\right)$ | $\mathbf{L}_{2}\left(\mathbf{L}_{\text {sun }}\right)$ | Spectral type $(\mathbf{V})$ | $\mathbf{M a s s}$ ratio $(\boldsymbol{\mu})$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 61 Cygni | 0.7000 | 0.6300 | 0.1530 | 0.0850 | K5/K7 | 0.4737 |
| Struve 2398 | 0.3340 | 0.2480 | 0.0390 | 0.0210 | M3/M3.5 |  |

Table 2. Showing the effect of oblateness up to zonal harmonic $J_{4}$ on the collinear points $\left(\boldsymbol{L}_{1,2,3}\right)$ for systems (61 Cygni and Struve 2398).

| Binary Systems | Mass ratio $(\boldsymbol{\mu})$ | $\boldsymbol{A}_{\mathbf{1}}$ | $\boldsymbol{B}_{\mathbf{1}}$ | $\boldsymbol{A}_{\mathbf{2}}$ | $\boldsymbol{B}_{\mathbf{2}}$ | $\boldsymbol{L}_{\mathbf{1}}$ | $\boldsymbol{L}_{\mathbf{3}}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 61 Cygni | 0.4737 | 0 | 0 | 0 | 0 | 0.82848 | 0.04357 | $\mathbf{- 1 . 8 4 6 5 1}$ |
|  |  | 0.01 | 0.02 | -0.005 | -0.01 | 0.86643 | 0.03870 | -2.28751 |
|  |  | 0.02 | 0.04 | -0.01 | -0.02 | 0.90523 | 0.03979 | -2.67807 |
| Struve 2398 | 0.03 | 0.06 | -0.015 | -0.03 | 0.94561 | 0.04683 | -3.04271 |  |
|  |  | 0.04 | 0.08 | -0.02 | -0.04 | 0.98836 | 0.06002 | -3.40197 |
|  |  | 0.4261 | 0.01 | 0 | 0 | 0 | 0.72743 | 0.10751 |

Graphs showing the effect of oblateness up to zonal harmonic $J_{4}$, the eccentricity and semi-major axis for the binary system; 61 Cygni.

(a)

(b)

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Figure 1. Effects of oblateness on $L_{1}$ with $\mu=0.4737, a=0.628, e=0.49, q_{1}=0.999767, q_{2}=0.999856$.


Figure 2. Effects of oblateness on $L_{2}$ with $\mu=0.4737, a=0.628, e=0.49, q_{1}=0.999767, q_{2}=0.999856$.



Figure 3. Effects of oblateness on $L_{3}$ with $\mu=0.4737, a=0.628, e=0.49, q_{1}=0.999767, q_{2}=0.999856$.
Graphs showing the effect of oblateness up to zonal harmonic $J_{4}$, the eccentricity and semi-major axis on the collinear points for the binary system Struve 2398.


Figure 4. Effects of oblateness on $L_{1}$ with $\mu=0.4261, a=0.2667, e=0.70, q_{1}=0.999876, q_{2}=0.9999098$.

(a)

(b)


Figure 5. Effects of oblateness on $L_{2}$ with $=0.4261, a=0.2667, e=0.70, q_{1}=0.999876, q_{2}=0.9999098$.


Figure 6. Effects of oblateness on $L_{3}$ with $=0.4261, a=0.2667, e=0.70, q_{1}=0.999876, q_{2}=0.9999098$.

## 6. Discussion and Conclusion

As shown in Equations (9), (12) and (15), the positions of collinear points are affected by the eccentricity, mass ratio, radiation pressure and oblateness up to zonal harmonic $J_{4}$ of the primaries. This agrees with the result of [14] with $\mathrm{J}_{4}=0$. The coordinates of the collinear equilibrium points are obtained numerically in table 2 and graphically in figures (1, 2, 3, 4, 5 and 6) for the binary systems: 61Cygni and Struve 2398. $L_{1}$ and $L_{2}$ increase with the increase in oblateness while the position of $L_{3}$ decreases with the increase in oblateness. This indicates that, the collinear points $L_{1}$ and $L_{2}$
move toward the less massive primary with increase in varying oblateness as shown in table 2 for the binary systems and figures ( $1,2,3,4,5$ and 6 ). Whereas, $L_{3}$ shifts closer to the more massive primary. By comparing the two binary systems, 61 Cygni and Struve 2398, it can be said that the effect of zonal harmonics does not show physically in the cases on the positions of $L_{1}$ and $L_{3}$ whereas it is shown on the positions of $L_{2}$ like in the case of 61 Cygni (Figure 2).

Figures (1-6) indicate that, the positions of collinear points do not move uniformly with increase in varying oblateness for the stated binary systems.

The stability of our collinear points affirms with those of [14]. As evidence in equations (18), (21) and (22), we have
$\Omega_{\xi \xi}^{0}>0, \Omega_{\eta \eta}^{0}<0$ and $\Omega_{\xi \eta}^{0}=0$ respectively for the collinear points lying in the interval $\left(\xi>\xi_{2}\right),\left(\xi_{1}<\xi<\xi_{2}\right)$ and $\left(\xi_{1}>\xi\right)$ with respect to their primaries, given that $\eta=\zeta=$ 0 . Since, $\Omega_{\xi \xi}^{0} \Omega_{\eta \eta}^{0}-\left(\Omega_{\xi \eta}^{0}\right)^{2}<0$ in equation (16), its discriminant is positive and the roots can be expressed as $\lambda_{1,2}= \pm a$ and $\lambda_{3,4}= \pm i b$ where $a$ and $b$ are real. This confirms that, the motion in the neighborhood of the collinear points is unstable since it is not bounded

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