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Ramanujan's Famous Partition Congruences

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Abstract

In 1742, firstly Leonhard Euler invented the generating function for *P*(n), where *P*(n) is the number of partitions of n [*P*(n) is defined to be 1]. Srinivasa Ramanujan was born on 22 December 1887. In 1916, S. Ramanujan invented the generating function for *P*(n) (2nd time). Godfrey Harold Hardy said Srinivasa Ramanujan was the first, and up to now the only, Mathematician to discover any such properties of *P*(n). MacMahon established a table of *P*(n) for the first 200 values of n, and Ramanujan observed that the table indicated certain simple congruences properties of *P*(n). In 1916, S. Ramanujan quoted his famous partition congrucnecs. In particular, the numbers of the partitions of numbers 5m+4, 7m+5, and 11m+6 are divisible by 5, 7, and 11 respectively. Now this paper shows how to prove the Ramanujan's famous partitions congruences modulo 5, 7, and 11 respectively.

Keywords

Congruences, Enumerating, Modulo, Residues, Ramanujan's Lost Notebook

1. Introduction

In this paper we give some related definitions of $P(n)$, Euler's product, Jacobi's product, Triangular number, Pentagonal number, $\lambda_1(n)$, $\lambda_2(n)$, and $\lambda_3(n)$. We discuss the generating functions for $\lambda_1(n)$, $\lambda_2(n)$ and $\lambda_3(n)$ respectively and prove the Ramanujan's famous partition congruences: $P(5m+4)\equiv 0(\bmod 5)$, $P(7m+5)\equiv 0(\bmod 7)$ and $P(11m+6)\equiv 0(\bmod 11)$ with the help of Euler's product, Jacobi's product and particular congruences [Ramanujan's Lost Notebook (1916)]

$$\frac{(1-x^5)\,(1-x^{10})\,(1-x^{15})\,\ldots\infty}{\left\{(1-x)\,(1-x^2)\,(1-x^3)\,\ldots\infty\right\}^5}\equiv 1(\bmod 5)$$

$$\frac{(1-x^7)\,(1-x^{14})\,(1-x^{21})\,\ldots\infty}{\left\{(1-x)\,(1-x^2)\,(1-x^3)\,\ldots\infty\right\}^7}\equiv 1(\bmod 7),\ \text{and}\ \frac{(1-x^{11})\,(1-x^{22})\,(1-x^{33})\,\ldots\infty}{\left\{(1-x)\,(1-x^2)\,(1-x^3)\,\ldots\infty\right\}^{11}}\equiv 1(\bmod 11).$$

2. Some Related Definitions

$P(n)$: The number of partitions of n like 4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore P(4)=5$.

Euler's product:

$$(1-x)(1-x^2)(1-x^3)..\ \infty = 1\text{-x-}x^2 + x^5 + x^7 - .... = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n-1)}{2}}$$

Jacobi's product:

$$\left\{(1-x)(1-x^2)(1-x^3)...\ \infty\right\}^3 = 1-3x+5x^3-7x^6+... = \sum_{n=0}^{\infty} (\text{-}1)^n (2n+1) x^{\frac{n(n+1)}{2}}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} (\text{-}1)^n (2n+1) x^{\frac{n(n+1)}{2}}$$

$\lambda_1(n)$: The number of partitions of n as the sum of integers, which are not multiples of 25.

$\lambda_2(n)$: The number of partitions of n as the sum of integers, which are not multiples of 49.

$\lambda_3(n)$: The number of partitions of n as the sum of integers, which are not multiples of 121.

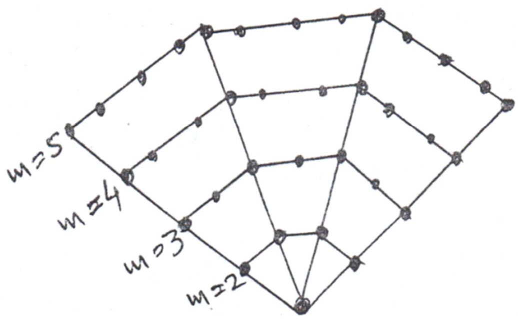
Triangular number: An integer that can be represented by a triangular array of dots like 1, 3, 6, 10, 15, 21, 28, ------- etc. or shortly $\frac{m(m+1)}{2}$; $m \in N$ (the set of natural integers). That is,

Triangular numbers

•	1
••	3
•••	6
••••	10
•••••	15
••••••	21
•••••••	28
....	etc.

Pentagonal numbers: The numbers $n = \frac{m(3m \pm 1)}{2}$ are called the pentagonal numbers like 1,5,12,,.. where $m \in N$ (the set of natural integers). If we consider a regular pentagon marked by 5 dots, so as to obtain successively pentagons with 3, 4, ..., ... dots on each side, then the total number of dots is $\frac{m(3m-1)}{2}$ and is shown by following figure 1,

Figure: Pentagonal Numbers



Pentagonal numbers : $n = \frac{m(3m-1)}{2}$

If m = 1 ;	n = 1
m = 2 ;	n = 5
m = 3 ;	n = 12
m = 4 ;	n = 22
m = 5 ;	n = 35
….	…

Figure 1. *The pentagonal numbers like 1,5,12,....*

3. The Generating Function for $\lambda_1(n)$ [S. Ramanujan (1916)] Is Given by

$$\frac{x(1-x^{25})\ (1-x^{50})\ (1-x^{75})\ ...\infty}{(1-x)\ (1-x^2)\ (1-x^3)\ ...\infty}$$

$$= \left\{\ x(1-x^{25})\ (1-x^{50})\ (1-x^{75})\ ...\infty\right\}\left\{\sum_{n=0}^{\infty} P(n) x^n\right\}$$

$$= \sum_{n=1}^{\infty} P(n)\ x^{n+1} - \sum_{n=0}^{\infty} P(n)\ x^{n+26} - \sum_{n=0}^{\infty} P(n)\ x^{n+51} + \sum_{n=0}^{\infty} P(n)\ x^{n+126} + ...\ \infty$$

$$= \sum_{n=1}^{\infty}\left[P(n-1) - P(n-26) - P(n-51) + P(n-126)\ + P(n-176) -\ ...\ \ \infty\right]\text{x}^{\text{n}}.$$

Where, 26, 51, 126, ... are numbers of the form $\frac{(5m-1)\ (15m-2)}{2}$ and $\frac{(5m+1)\ (15m+2)}{2}$, such that m is any positive integer. Therefore,

$$\frac{x\left(1-x^{25}\right)\left(1-x^{50}\right)\left(1-x^{75}\right)\ldots\infty}{(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right)\ldots\infty}$$

$$=\sum_{n=1}^{\infty}\Big[P(n-1)+\sum_{m=1}^{\infty}(-1)^{m}P\left(n-\frac{(5m-1)(15m-2)}{2}\right)+\sum_{m=1}^{\infty}(-1)^{m}P\left(n-\frac{(5m+1)(15m+2)}{2}\right)\Big]x^{n}=\sum_{n=1}^{\infty}\lambda_1(n)x^{n},$$

where, $\lambda_1(n)=P(n-1)+\sum_{m=1}^{\infty}(-1)^{m}P\left(n-\frac{(5m-1)(15m-2)}{2}\right)+\sum_{m=1}^{\infty}(-1)^{m}P\left(n-\frac{(5m+1)(15m+2)}{2}\right)$.

So the coefficient $\lambda_1(n)$ of x^n in the above expansion is the number of partitions of *n* as the sum of integers, which are not multiples of 25.

It shows that the coefficient of x^{5n} in the above expansion is a multiple of 5 and we follow that $P(4)$, $P(9)$, $P(14)$, … $\equiv 0(\bmod 5)$, i.e., $P(5m+4)\equiv 0(\bmod 5)$, where *m* is a non-negative integer.

Now it needs to be proved.

Theorem 1.1: $P(5m+4)\equiv 0(\bmod 5)$

Proof: We get;

$$x\left\{(1-x)\left(1-x^{2}\right)\ldots\infty\right\}^{4}=x\left\{(1-x)\left(1-x^{2}\right)\ldots\infty\right\}^{3}\left\{(1-x)\left(1-x^{2}\right)\ldots\infty\right\}$$

$$=x(1-3x+5x^{3}-7x^{6}+\ldots)(1\text{-x-x}^{2}+\text{x}^{5}+\ldots)$$

$$=\frac{1}{2}\sum_{\mu=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\nu+1)x^{1+\frac{1}{2}\mu(3\mu+1)+\frac{1}{2}\nu(\nu+1)}$$ [By Euler and Jacobi's identities]… (1)

$$=\frac{1}{2}\sum_{\mu=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\nu+1)x^{k}, \text{ where}$$

$$k=k(\mu,\nu)=1+\frac{1}{2}\mu(3\mu+1)+\frac{1}{2}\nu(\nu+1).$$

We consider the index of *x* is divisible by 5, and the index

$$k=1+\frac{1}{2}\mu(3\mu+1)+\frac{1}{2}\nu(\nu+1)$$

$8k=12\mu^{2}+4\nu^{2}+4(\mu+\nu)+8$. Now,

$2(\mu+1)^{2}+(2\nu+1)^{2}=2\mu^{2}+4\mu+2+4\nu^{2}+4\nu+1$

$=8k-10\mu^{2}-5$ is also divisible by 5.We have;

$2(\mu+1)^{2}\equiv 0,2 \text{ or } 3(\bmod 5)$ and $(2\nu+1)^{2}\equiv 0,1 \text{ or } 4(\bmod 5)$, we see by enumerating the possible cases, if $\mu\equiv 4(\bmod 5)$ and $\nu\equiv 2(\bmod 5)$, then the sum of residues from above two congruences is 0, consequently the every term is 0 modulo 5. Hence if the index *k* is divisible by 5, the coefficient $(2\nu+1)$ in (1.1) is also divisible by 5, and therefore the coefficient of x^{5m+5} in $x\left\{(1-x)\left(1-x^{2}\right)\ldots\infty\right\}^{4}$ is divisible by 5.

Hence the coefficient of x^{5m+5} in $\frac{x\left(1-x^{5}\right)\left(1-x^{10}\right)\ldots\infty}{(1-x)\left(1-x^{2}\right)\ldots\infty}$

$$=x\left\{(1-x)\left(1-x^{2}\right)\ldots\infty\right\}^{4}\frac{\left(1-x^{5}\right)\left(1-x^{10}\right)\ldots\infty}{\left\{(1-x)\left(1-x^{2}\right)\ldots\infty\right\}^{5}}$$ is a multiple of 5.

Since, $\frac{\left(1-x^{5}\right)\left(1-x^{10}\right)\left(1-x^{15}\right)\ldots\infty}{\left\{(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right)\ldots\infty\right\}^{5}}\equiv 1(\bmod 5)$.

But $$\frac{x}{(1-x)\left(1-x^{2}\right)\ldots\infty}=\frac{x\left(1-x^{5}\right)\left(1-x^{10}\right)\ldots\infty}{(1-x)\left(1-x^{2}\right)\ldots\infty}\left(1+x^{5}+x^{10}+\ldots\infty\right)\left(1+x^{10}+x^{20}+\ldots\infty\right)\ldots$$

Hence the coefficient of x^{5m+5} in

$$\frac{x}{(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right)\ldots\infty}=x+\sum_{n=2}^{\infty}P(n-1)x^{n}$$ is divisible by 5.

Thus $P(5m+4)\equiv 0(\bmod 5)$. Hence the Theorem.

Example 1.1: We get; $p(4)=5,\ p(9)=30,\ p(14)=135,\ \ldots$

$i,e,\ p(4)=5\equiv 0\ (\bmod 5),\ p(9)=30\equiv 0\ (\bmod 5),\ p(14)=135\equiv 0\ (\bmod 5),\ \ldots$

We can conclude that $P(5m+4)\equiv 0(\bmod 5)$, where m is a non-negative integer.

4. The Generating Function for $\lambda_2(n)$ [S. Ramanujan (1916)] Is Given By

$$\frac{x^{2}(1-x^{49})(1-x^{98})(1-x^{147})\ldots\infty}{(1-x)(1-x^{2})(1-x^{3})(1-x^{4})\ldots\infty}=\left\{x^{2}(1-x^{49})(1-x^{98})(1-x^{147})\ldots\infty\right\}\left\{\sum_{n=0}^{\infty}p(n)x^{n}\right\}$$

$$=\sum_{n=0}^{\infty}P(n)\,x^{n+2}-\sum_{n=0}^{\infty}P(n)\,x^{n+51}-\sum_{n=0}^{\infty}P(n)\,x^{n+100}+\sum_{n=0}^{\infty}P(n)\,x^{n+247}+\ldots\ \infty$$

$$=\sum_{n=2}^{\infty}\left[P(n-2)-P(n-51)-P(n-100)+P(n-247)+P(n-345)-\ldots\infty\right]x^n,$$

where 51, 100, 247, 345, ... , are numbers of the form $\frac{(7m-1)\ (21m-4)}{2}$ and $\frac{(7m+1)\ (21m+4)}{2}$, such that m is any positive integer. Therefore,

$$\frac{x^2(1-x^{49})(1-x^{98})(1-x^{147})\ldots\infty}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\infty}$$

$$=\sum_{n=2}^{\infty}\left[P(n-2)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(7m-1)\ (21m-4)}{2}\right)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(7m+1)\ (21m+4)}{2}\right)\right]x^n=\sum_{n=2}^{\infty}\lambda_2(n)x^n,$$

where, $\lambda_2(n)=P(n-2)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(7m-1)\ (21m-4)}{2}\right)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(7m+1)\ (21m+4)}{2}\right).$

So the coefficient $\lambda_2(n)$ of x^n in the above expansion is the number of partitions of n as the sum of integers, which are not multiples of 49,

It shows that the coefficient of x^{7n} in the above expansion is a multiple of 7. We follow that $P(5)$, $P(12)$, $P(19)$, … $\equiv 0(\bmod 7)$, i.e., $P(7m+5)\equiv 0(\bmod 7)$, where m is a non-negative integer.

Theorem 1.2 [Ramanujan (1916)]: $P(7m+5)\equiv 0(\bmod 7)$

Proof: We get;

$$x^2\left\{(1-x)\ (1-x^2)\ldots\infty\right\}^6=x^2\left\{(1-x)\ (1-x^2)\ldots\infty\right\}^3\left\{(1-x)\ (1-x^2)\ldots\infty\right\}^3$$

$$=\frac{1}{4}x^2\sum_{\mu=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\mu+1)\ (2\nu+1)x^{\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)}$$ [By Jacobi's identity]

$$=\frac{1}{4}\sum_{\mu=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\mu+1)\ (2\nu+1)x^{2+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)}$$

$$=\frac{1}{4}\sum_{\mu=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\mu+1)\ (2\nu+1)x^{k}. \tag{2}$$

Where $k=2+\frac{1}{2}\mu\ (\mu+1)+\frac{1}{2}\nu(\nu+1)$

$2k=\mu^2+\nu^2+\mu+\nu+4$

$8k=4(\mu^2+\nu^2)+4(\mu+\nu)+16$.

Again, $(2\mu+1)^2+(2\nu+1)^2=4(\mu^2+\nu^2)+4(\mu+\nu)+2$

$=8k-14$, which can be divisible by 7 only if the index k is divisible by 7.

Now we have;

$(2\mu+1)^2\equiv 0,1,2$ or $4(\bmod 7)$ and

$(2\nu+1)^2\equiv 0,1,2$ or $4(\bmod 7)$.

We can see by enumerating the possible cases if $\mu\equiv 3(\bmod 7)$ and $\nu\equiv 3(\bmod 7)$, then the sum of residues from above congruences is 0. Therefore the every term is 0 modulo 7. Hence, if the index in (1.2) is a multiple of 7, the coefficient $\{(2\mu+1).(2\nu+1)\}$ is also multiple of 7, and such that m is any positive integer. Therefore, therefore the coefficient of x^{7m+7} in $x^2\left\{(1-x)\ (1-x^2)\ (1-x^3)\ldots\infty\right\}^6$ is a multiple of 7.

Hence we can easily verify that the coefficient of x^{7m+7} in

$$\frac{x^2(1-x^7)\ (1-x^{14})\ \ldots\infty}{(1-x)\ (1-x^2)\ \ldots\infty}$$

$$=x^2\left\{(1-x)\ (1-x^2)\ \ldots\infty\right\}^6\frac{(1-x^7)\ (1-x^{14})\ \ldots\infty}{\left\{(1-x)\ (1-x^2)\ \ldots\infty\right\}^7}$$

is a multiple of 7. So the coefficient of x^{7m+7} in

$$\frac{x^2}{(1-x)\ (1-x^2)\ (1-x^3)\ \ldots\infty}=x^2+\sum_{n=3}^{\infty}P(n-2)x^n$$ is a multiple of 7.

Thus $P(7m+5)\equiv 0(\bmod 7)$. Hence the Theorem

Example 1.2: We get; $p(5)=7,\ p(12)=77,\ \ldots$

$i,e,\ p(5)=7\equiv 0\ (\bmod 7),\ p(12)=77\equiv 0\ (\bmod 7),\ \ldots$

We can conclude that $P(7m+5)\equiv 0(\bmod 7)$, where m is a non-negative integer.

5. The Generating Function for $\lambda_3(n)$ [S. Ramanujan (1916)] Is Given by

$$\frac{x^5(1-x^{121})(1-x^{242})(1-x^{363})\ldots\infty}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\infty}$$

$$=x^5\left\{(1-x^{121})(1-x^{242})(1-x^{363})\ldots\infty\right\}\left\{\sum_{n=0}^{\infty}P(n)x^n\right\}$$

$$=\sum_{n=0}^{\infty}P(n)\ x^{n+5}-\sum_{n=0}^{\infty}P(n)\ x^{n+126}-\sum_{n=0}^{\infty}P(n)\ x^{n+247}+\sum_{n=0}^{\infty}P(n)\ x^{n+610}+\ldots\ \infty$$

$$=\sum_{n=5}^{\infty}\left[P(n-5)-P(n-126)-P(n-247)+P(n-610)+P(n-852)-\ldots\infty\right]x^n,$$

where 126, 247, 610, 852, ... are numbers of the form $\frac{(11m-2)\ (33m-5)}{2}$ and $\frac{(11m+2)\ (33m+5)}{2}$,

$$\frac{x^5(1-x^{121})(1-x^{242})(1-x^{363})\ldots\infty}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\infty}$$

$$=\sum_{n=5}^{\infty}\left[P(n-5)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(11m-2)\ (33m-5)}{2}\right)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(11m+2)\ (33m+5)}{2}\right)x^n\right.$$

$$=\sum_{n=5}^{\infty}\lambda_3(n)x^n\ .$$

Where, $\lambda_3(n)=P(n-5)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(11m-2)\ (33m-5)}{2}\right)+\sum_{m=1}^{\infty}(-1)^m P\left(n-\frac{(11m+2)\ (33m+5)}{2}\right).$

So the coefficient $\lambda_3(n)$ of x^n in the above expansion is the number of partitions of *n* as the sum of integers, which are not multiples of 121,

It shows that the coefficient of x^{11n} in the above expansion is a multiple of 11. We follow that $P(6)$, $P(17)$, $P(28),\ldots \equiv 0(\bmod 11)$, i.e., $P(11m+6)\equiv 0(\bmod 11)$, where *m* is a non-negative integer.

Now we prove the Theorem.

Theorem 1.3: $P(11m+6)\equiv 0(\bmod 11)$

Proof: We get; $x^5\left\{\ (1-x)\ \left(1-x^2\right)\ldots\infty\right\}^{10}$

$$=x^5\left\{\ (1-x)\ \left(1-x^2\right)\ldots\infty\right\}^3\left\{\ (1-x)\ \left(1-x^2\right)\ldots\infty\right\}^3\left\{\ (1-x)\ \left(1-x^2\right)\ldots\infty\right\}^3\left\{\ (1-x)\ \left(1-x^2\right)\ldots\infty\right\}$$

$$=\frac{1}{8}x^5\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\sum_{m_3=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}(-1)^{m_1+m_2+m_3+r}(2m_1+1)(2m_2+1)(2m_3+1)x^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{r(3r+1)}{2}}$$

[By Euler and Jacobi's identities]

$$=\frac{1}{8}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\sum_{m_3=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}(-1)^{m_1+m_2+m_3+r}(2m_1+1)(2m_2+1)(2m_3+1)x^{5+\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{r(3r+1)}{2}}$$

$$=\frac{1}{8}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\sum_{m_3=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}(-1)^{m_1+m_2+m_3+r}(2m_1+1)(2m_2+1)(2m_3+1)x^k. \tag{3}$$

Where, $k=5+\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{r(3r+1)}{2}$

$2k=m_1^2+m_2^2+m_3^2+m_1+m_2+m_3+3r^2+r+10$

$8k=4\left(m_1^2+m_2^2+m_3^2+m_1+m_2+m_3+r\right)+12r^2+40$.

We have; $(2m_1+1)^2+\ (2m_2+1)^2+\ (2m_3+1)^2+(r+2)^2$

$=4\left(m_1^2+m_2^2+m_3^2+m_1+m_2+m_3+r\right)+r^2+7$

$=8k-11r^2-33$, which can be divisible by 11 only if the index *k* is divisible by 11. Now,

$(2m_1+1)^2\equiv 0,1,3,4,5 \text{ or } 9(\bmod 11)$

$(2m_2+1)^2\equiv 0,1,3,4,5 \text{ or } 9(\bmod 11)$

$(2m_3+1)^2\equiv 0,1,3,4,5 \text{ or } 9(\bmod 11)$ and

$(r+2)^2\equiv 0,1,3,4,5 \text{ or } 9(\bmod 11)$.

We can see by enumerating the possible cases if $m_1\equiv 5(\bmod 11)$, $m_2\equiv 5(\bmod 11)$, $m_3\equiv 5(\bmod 11)$ and $r\equiv 5(\bmod 11)$, then the sum of residues from above congruences is 0. Consequently the every term is 0 modulo 11. Hence if the index *k* in (1.3) is divisible by 11, the coefficient $\{(2m_1+1)\ (2m_2+1)\ (2m_3+1)\}$ is also divisible by 11, and therefore the coefficient of x^{11m+11} in $x^5\left\{(1-x)\ \left(1-x^2\right)\ldots\infty\right\}^{10}$ is divisible by 11.

Hence we can easily verify that the coefficient of x^{11m+11} in

$$\frac{x^5\left(1-x^{11}\right)\ \left(1-x^{22}\right)\ \ldots\infty}{(1-x)\ \left(1-x^2\right)\ \ldots\infty}=x^5\left\{(1-x)\ \left(1-x^2\right)\ \ldots\infty\right\}^{10}\frac{\left(1-x^{11}\right)\ \left(1-x^{22}\right)\ \ldots\infty}{\left\{(1-x)\ \left(1-x^2\right)\ \ldots\infty\right\}^{11}}$$

is a multiple of 11.

Finally, the coefficient of x^{11m+11} in

$$\frac{x^5}{(1-x)\ (1-x^2)\ \ldots\infty} = x^5 + \sum_{n=6}^{\infty} P(n-5)\ x^n$$

is a multiple of 11. Thus $P(11m+6) \equiv 0(\bmod 11)$. Hence the theorem

Example 1.3: We get; $p(6)=11,\ \ p(17)=297,\ \ \ldots$

$i,e,\ p(6)=11 \equiv 0\ (\bmod\ 11),\ p(17)=297 \equiv 0\ (\bmod\ 11),\ \ \ldots$

We can conclude that $P(11m+6) \equiv 0(\bmod 11)$, where m is a non-negative integer.

6. Conclusion

In this paper we have defined the Triangular number, Pentagonal numbers with the help of figures. For any non-negative integer of m we have proved the theorem 1.1and have verified the Theorem with m = 0,1,... We have established the Theorem 1.2 and have verified the Theorem with m=0,1,… Finally we have proved the Theorem 1.3 by algebraic method and have verified the Theorem with m=0,1..., respectively.

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