# Bayesian Analysis of Paired Comparison Model using Jeffreys Prior 

Amna Nazeer ${ }^{1, *}$, Sadia Qamar ${ }^{2}$, Samina Satti ${ }^{3}$<br>${ }^{1}$ School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, China<br>${ }^{2}$ Department of Statistics, University of Sargodha, Sargodha, Pakistan<br>${ }^{3}$ Department of Statistics, University of Wah, WahCantt, Pakistan<br>Email address<br>amna.stats@yahoo.com (A. Nazeer)

## To cite this article

Amna Nazeer, Sadia Qamar, Samina Satti. Bayesian Analysis of Paired Comparison Model using Jeffreys Prior. Open Science Journal of Statistics and Application. Vol. 3, No. 6, 2015, pp. 38-41.


#### Abstract

Paired comparison is very old and reliable psychometric scheme. Baaren in 1978 presented new extensions of the paired comparison models. This study contains the Bayesian analysis of the Baaren model-IV using non-informative Jeffreys prior. The paired comparison model includes the treatment/worth, tie and within pair order effect parameters. Four treatments are used for the numerical evaluation of the model. Due to the complex description of the Jeffreys prior for the current study, it has been approximated numerically. Gibbs sampling method has been used for the approximation of the findings. The joint posterior distribution was then obtained and used to compute the posterior means, posterior modes and posterior standard deviations. The findings supported the existence of the order effect i.e. the treatment presented first had an edge of being preferred in the pair wise comparison. The preference and posterior probabilities of the model also supported the findings of the posterior estimates. The $\mathrm{X}^{2}$ test declared the model appropriate for the under study data set with high probability.


## Keywords

Paired Comparison Model, Bayesian Inference, Jeffreys Prior, Posterior Estimates, Preference Probabilities

## 1. Introduction

The paired comparison method is very old and well-developed ranking technique that has been used by several generations of psychologists, pharmacist, agronomist etc. The method simply comprises of pairing the items/treatments in hand with each of the other items. The pairs are then presented to the judges and every response is simply a choice between those two items in pair. The judges under some situations are also allowed to state their preference on basis of some scale of preference or remain indifference [1]. The method of paired comparison is a simple and less expensive than other methods and it may be regarded as a special rank order technique, which provides an efficient tool for the assessment of preference [2]. The very basic and simple paired comparison model known as Bradley-Terry (B-T) paired comparison model [3] only allows the preference of the either object under comparison. The B-T model has further been modified to develop new paired comparison models by adjusting tie parameter $[4,5]$ and with-in pair order
of presentation effect parameter [6]. Under different substitutions of parameters, [7] has also added new extensions of the B-T paired comparison model. Many researcher [8-13] among others have explored, generalized and widen the scope of paired comparison approach. The current study has focused on the extensions of the Baaren paired comparison model viz. model-IV and explored it using Bayesian inferential methods. Jeffreys non-informative prior has been used for this purpose. [14-19] have conducted Bayesian analysis of the paired comparison models in very comprehensive way. The article is designed as: section 2 contains the details of the model; section 3 explains the Bayesian study of the model, section 4 checks the fitness of the model for the under study data followed by the conclusion in the section 5 .

## 2. The Model and the Data

Suppose a paired comparison experiment with $t$ treatments/items to be presented to the judges in pairs who, considering the understudy characteristic, state their response in favour for one of the two under comparison items or state a
tie. With $\pi_{i}$ as worth/preference parameter of treatment ' $i$ ', $\gamma$ as with-in pair order effect parameter and $v$ as the tie or threshold parameter the model is defined as (for details see [7]):

$$
\begin{align*}
& \pi_{i, i j}=\frac{\gamma \pi_{i}}{\gamma \pi_{i}+\pi_{j}+v\left(1+t \sqrt{\gamma \pi_{i} \pi_{j}}\right)} \\
& \pi_{j . i j}=\frac{\pi_{j}}{\gamma \pi_{i}+\pi_{j}+v\left(1+t \sqrt{\gamma \pi_{i} \pi_{j}}\right)}  \tag{1}\\
& \pi_{o . i j}=\frac{v\left(1+t \sqrt{\gamma \pi_{i} \pi_{j}}\right)}{\gamma \pi_{i}+\pi_{j}+v\left(1+t \sqrt{\gamma \pi_{i} \pi_{j}}\right)}
\end{align*}
$$

Where $(i, j=1,2, \ldots, t ; i \neq j)$ and the probabilities define in expression (1) sum to 1 .

With $v$ approach 0 and $\gamma$ approach 1 , the model-IV defined through equation 1 becomes equivalent to the B-T model. Taking into account the situation when the judges are to state their preferences regarding four treatments i.e. for ' $t=4$ ', with four worth parameters $\pi_{1}, \pi_{2}, \pi_{3}$ and $\pi_{4}$ that measure the true worth of the under study four treatments respectively. The likelihood function for model is given as:

$$
\begin{array}{r}
l\left(\mathrm{x} ; \pi_{1}, . ., \pi_{\mathrm{t}}, v, \gamma\right)= \\
\gamma^{\mathrm{n}} \prod_{\mathrm{i} \neq \mathrm{j}=1}^{\mathrm{t}} \mathrm{~K}_{\mathrm{ij}} \pi_{\mathrm{i}}^{\mathrm{n}_{\mathrm{i}}}\left(v\left(1+\mathrm{t} \sqrt{\gamma \pi_{\mathrm{i}} \pi_{\mathrm{j}}}\right)\right)^{\mathrm{n}_{0}} \mathrm{~A}_{\mathrm{ij}} \mathrm{n}_{\mathrm{ij}} \tag{2}
\end{array}
$$

Where $\quad \pi_{i} \geq 0 \quad, \quad \sum_{i=1}^{t} \pi_{i}=1 \quad, \quad A_{i j}=\gamma \pi_{i}+\pi_{j}+$ $v\left(1+t \sqrt{\gamma \pi_{i} \pi_{j}}\right), K_{i j}=\frac{n_{i j}!}{\left(n_{0, i j}!n_{i, i j}!n_{j i j}!!\right.}$ and $v, \gamma \geq 0$.
$\mathrm{n}_{\mathrm{i} . \mathrm{ij}}(\mathrm{i}, \mathrm{j}=1,2,3,4 ; \mathrm{i} \neq \mathrm{j})$ is the number of times treatment ' $i$ ' is preferred to treatment ' $j$ ' when treatment ' $i$ ' is presented first. $n_{i}$ is total number of times the treatment $i$ is preferred to any other treatment, $n_{0}$ is the total number of ties, $n$ represents the number that the treatment presented first is preferred and $n_{i j}$ is the total number of comparisons in the experiment. The data for the analysis that allows order effect has been taken from [20] that comprises of 30 respondents who were asked to report their preference about four products presented to them in pairs. The frequencies of their preferences and ties are given in Table 1.

Table 1. Data for Tie and Order Effect with $t=4$.

| Pairs | $\boldsymbol{n}_{\mathbf{i j}}$ | $\boldsymbol{n}_{\mathbf{i} . \mathrm{ij}}$ | $\boldsymbol{n}_{\mathbf{j} . \mathrm{ij}}$ | $\boldsymbol{n}_{\mathbf{0} \mathbf{i j}}$ |
| :--- | :--- | :--- | :--- | :--- |
| $(1,2)$ | 30 | 17 | 10 | 3 |
| $(2,1)$ | 30 | 22 | 6 | 2 |
| $(1,3)$ | 30 | 21 | 7 | 2 |
| $(3,1)$ | 30 | 18 | 9 | 3 |
| $(1,4)$ | 30 | 22 | 7 | 1 |
| $(4,1)$ | 30 | 16 | 12 | 2 |
| $(2,3)$ | 30 | 23 | 6 | 1 |
| $(3,2)$ | 30 | 17 | 10 | 3 |
| $(2,4)$ | 30 | 22 | 6 | 2 |
| $(4,2)$ | 30 | 17 | 11 | 2 |
| $(3,4)$ | 30 | 20 | 7 | 3 |
| $(4,3)$ | 30 | 19 | 8 | 3 |

[^0]
## 3. Bayesian Analysis of the Model Using Jeffreys Prior

Being invariant under transformation, Jeffreys prior is the most widely used prior in the Bayesian analysis. It is most appropriate in the situations where the under study model has no nuisance parameters and also the posterior distribution is asymptotically normal [21]. It can easily be seen from equation 2 that the joint distribution of the worth parameters is multinomial distribution hence the likelihood function belongs to the exponential family and so the posterior distribution will be asymptotically normal and also there are no nuisance parameters in the model so Jeffreys prior is an proper candidate choice for the analysis. Jeffreys prior is defined as the square root of the Fisher's Information matrix. For the given case it is expressed in the following form:

$$
\begin{equation*}
\mathrm{P}(\psi) \propto \sqrt{\operatorname{detI}(\psi)} \tag{3}
\end{equation*}
$$

Where $\psi=\left\{\pi_{1}, \pi_{2}, \pi_{3}, \gamma, v\right\}$ and $\mathrm{I}(\psi)$ denotes the $\mathrm{t} \times \mathrm{t}$ Fisher's information matrix and is given below:

$$
\mathrm{I}_{\mathrm{ij}}(\psi)=-\mathrm{E}\left[\frac{\partial^{2} \mathrm{~L}(\psi)}{\partial \psi_{\mathrm{i}} \partial \psi_{\mathrm{j}}}\right]
$$

Where ' $E$ ' stands for expectation on data and $i, j$ stands for rows and columns of determinant and $\mathrm{L}(\psi)$ is the logarithm of the likelihood function. Due to the complex mathematical form of the determinants, the Jeffreys prior is approximated numerically by designing computer program in SAS software and for further Bayesian analysis the numerical form of the Jeffreys is used.

The joint posterior distribution of four treatment treatments that includes six parameters i.e. $\pi_{1}, \pi_{2}, \pi_{3}, \pi_{4}, v$ and $\gamma$ using Jeffreys prior given in equation 3 is given as:

$$
\begin{align*}
& p\left(\pi_{1}, \pi_{2}, \pi_{3}, v, \gamma \mid x\right)=K^{-1} p_{j}\left(\pi_{1}, \pi_{2}, \pi_{3}, \gamma, v\right) \gamma^{n} \pi_{1}^{n_{1}} \pi_{2}^{n_{2}} \pi_{3}^{n_{3}}\left(1-\pi_{1}-\pi_{2}-\pi_{3}\right)^{n_{4}} \\
& \prod_{j \neq i=1}^{4}\left\{v\left(1+4 \sqrt{\gamma \pi_{i} \pi_{j}}\right)\right\}^{n_{0}} A_{i j}^{-n_{y}} \tag{4}
\end{align*}
$$

where $\pi_{i} \geq 0, \sum_{i=1}^{t} \pi_{i}=1, v, \gamma \geq 0$ and $p_{j}\left(\pi_{1}, \pi_{2}, \pi_{3}, v, \gamma\right)$ is the Jeffreys prior distribution and K represents the normalizing constant. Using the numerical form of Jeffreys prior; the posterior estimates of the parameters are computed. The posterior means, S.D.s and CVs for the parameters of the model are computed using Gibbs sampling method of numerical approximation and the results are provided in Table 2.

Table 2. Statistical Summary of the parameters.

| Parameters | $\boldsymbol{\pi}_{\mathbf{1}}$ | $\boldsymbol{\pi}_{\mathbf{2}}$ | $\boldsymbol{\pi}_{\mathbf{3}}$ | $\boldsymbol{\pi}_{\mathbf{4}}$ | $\boldsymbol{\gamma}$ | $\boldsymbol{v}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Mean | 0.2608 | 0.3410 | 0.2071 | 0.1920 | 2.6867 | 0.0421 |
| S.D. | 0.0355 | 0.0485 | 0.0332 | 0.0381 | 0.3064 | 0.0002 |
| CV | 13.625 | 14.212 | 16.056 | 19.824 | 11.405 | 0.4039 |

It is clear from the table that the ranking of the treatment 2 is placed at top, treatment 1 and 3 at second and third respectively while treatment 4 is placed last. Further the estimated value for the order effect parameter is greater than 1 showing the treatment presented first will get benefit. The
posterior modes of the parameters $\pi_{1}, \pi_{2}, \pi_{3}, \pi_{4}, \nu$ and $\gamma$ are computed as $0.26239,0.33994,0.19194,2.56426$ and 0.04239 . On comparing these results with posterior means, it is clear that joint posterior modes are very similar to the posterior means of their corresponding parameters. Hence ranking of the treatments remains same by the both estimates.

By utilizing the posterior means of parameters as posterior estimates, the preference probabilities (Table 3) are calculated by substituting the posterior means in the model given in equation 1 .

Table 3. Preference probabilities using Jeffreys prior.

| Pairs $(\mathbf{i}, \mathbf{j})$ | $\boldsymbol{\pi}_{\boldsymbol{i} \boldsymbol{i} \boldsymbol{j}}$ | $\boldsymbol{\pi}_{\boldsymbol{j} \boldsymbol{i j} \boldsymbol{j}}$ | $\boldsymbol{\pi}_{\mathbf{0 . \boldsymbol { i } \boldsymbol { j }}}$ |
| :--- | :--- | :--- | :--- |
| $(1,2)$ | 0.5876 | 0.3045 | 0.1079 |
| $(2,1)$ | 0.7137 | 0.1919 | 0.0944 |
| $(1,3)$ | 0.6822 | 0.2113 | 0.1065 |
| $(3,1)$ | 0.6406 | 0.2746 | 0.1484 |
| $(1,4)$ | 0.6983 | 0.1959 | 0.1058 |
| $(4,1)$ | 0.5888 | 0.2924 | 0.1188 |
| $(2,3)$ | 0.7420 | 0.1656 | 0.0924 |
| $(3,2)$ | 0.5457 | 0.3407 | 0.1136 |
| $(2,4)$ | 0.7561 | 0.1528 | 0.0912 |
| $(4,2)$ | 0.5229 | 0.3606 | 0.1165 |
| $(3,4)$ | 0.6619 | 0.2236 | 0.1145 |
| $(4,3)$ | 0.6237 | 0.2571 | 0.1192 |

The results of the preference probabilities are according to the findings of the posterior estimates. It can clearly be seen that the preferences are under the influence of order as for each comparison the probability for the treatment presented first is higher.

Consider the following hypotheses for the data set given in Table 1 for the comparison of treatment pair ' $i$ ' and ' $j$ '.
$\mathrm{H}_{\mathrm{ij}}: \pi_{\mathrm{i}}>\pi_{\mathrm{j}}$ and $\mathrm{H}_{\mathrm{ij}}: \pi_{\mathrm{j}} \geq \pi_{\mathrm{i}}$. The posterior probability $H_{i j}=p_{i j}=p\left(\pi_{i}>\pi_{j}\right)$ for the model can be determined in terms of marginal posterior distribution, which can be calculated from the joint posterior distribution of $\varphi=\left(\pi_{i}-\right.$ $\pi_{j}$ ) and $\varepsilon=\pi_{i}$. The posterior probability for $H_{j i}$ is given by $q_{i j}=1-p_{i j}$.

The decision rule used here for accepting or rejecting the above hypothesis is

$$
\begin{equation*}
\text { Let } s=\min \left(\mathrm{p}_{\mathrm{ij}}, \mathrm{q}_{\mathrm{ij}}\right), \tag{5}
\end{equation*}
$$

If $p_{i j}$ is small then $H_{j i}$ is accepted, if $q_{i j}$ is small then $H_{i j}$ is accepted and the decision is inconclusive when $s>0.1$ [16]. The terms $\mathrm{p}_{\mathrm{ij}}$ and $\mathrm{q}_{\mathrm{ij}}$ measure the posterior probabilities for the hypotheses $\mathrm{H}_{\mathrm{ij}}$ and $\mathrm{H}_{\mathrm{ji}}$ concerning the comparison of two treatment parameters and Table 4 contains the results of posterior probabilities.

Table 4. Posterior probabilities.

| Pairs $(\mathbf{i}, \mathbf{j})$ | $\boldsymbol{p}_{\boldsymbol{i j}}$ | $\boldsymbol{q}_{\boldsymbol{i j}}$ |
| :--- | :--- | :--- |
| $(1,2)$ | 0.0867 | 0.9133 |
| $(1,3)$ | 0.8511 | 0.1489 |
| $(1,4)$ | 0.8994 | 0.1006 |
| $(2,3)$ | 0.9778 | 0.0222 |
| $(2,4)$ | 0.9844 | 0.0156 |
| $(3,4)$ | 0.6156 | 0.3844 |

The hypotheses $H_{12}, H_{23}$ and $H_{24}$ are accepted with high probabilities using both methods. The hypotheses $H_{13}, H_{34}$ and $H_{14}$ are shown inconclusive due to insufficient evidence.

## 4. Appropriateness of the Model

Appropriateness of the model is tested by comparing the observed and the expected number of preferences. The $\chi^{2}$ statistic given in [20] is employed to test the fitness of the model. For testing, we define the hypothesis as:
$H_{0}$ : Model is appropriate for some values of $\pi$.
$H_{1}$ : Model is not fit for any value of $\pi$.
The $\chi^{2}$ statistic is found to be 9.84904 with p -value as 0.956567 so clearly, there is no evidence that the model does not fit.

## 5. Conclusion

Paired comparison is a very straightforward ranking method that has been in application from decades. The current study has attempted to analyze the paired comparison model proposed by Baaren (1978) using the Bayesian paradigm. The analysis provide a very refined scenario of ranking of the under study treatments. The findings of the posterior estimates (mean and mode) have helped to rank the treatment 2 as the highly preferred treatment. Treatment 4 is the least preferred and treatments 1 and 3 share the second and the third place respectively. The effect of the order of presentation is visible in the study and same can be observed from the findings of the preference probabilities. The posterior probabilities have also supported the findings of the posterior estimates and probability of preferring treatment 2 in comparison with any other treatment is always found to be higher. The $\chi^{2}$ statistic has supported the model for the under consideration data set. The $p$-value is found to be far greater than 0.05 . So the model has appeared to be a better fit for the data of paired comparison experiments.

## References

[1] David, H. A., (1988). The Method of Paired Comparisons. $2^{\text {nd }}$ Eds. London: Griffin.
[2] Bradley, R. A. (1976). A biometrics invited paper. science, statistics, and paired comparisons. Biometrics, 32, 213-239.
[3] Bradley, R. A., \& Terry, M. E. (1952). Rank analysis of Incomplete Block Designs, I. The method of Paired Comparisons. Biometrika, 39, 324-345.
[4] Rao, P. V., \& Kupper, L. L. (1967). Ties in Paired-Comparison Experiments: A Generalization of Bradley-Terry Model. Journal of the American Statistical Association, 62, 194-204.
[5] Davidson, R. R. (1970). On Extending the Bradley-Terry Model to Accommodate Ties in the Paired comparison Experiments. Journal of the American Statistical Association, 65, 317-328.
[6] Davidson, R. R., \& Beaver, R. J. (1977). On Extending the Bradley-Terry Model to Incorporate Within Pair Order Effects. Biometrics, 33, 693-702.
[7] Baaren, A. V. (1978). On a Class of Extension to the Bradley-terry Model in Paired Comparisons. Statistica Neerlandica, 32, 57-67.
[8] Trawinski, B. J. (1965). An Exact Probability Distribution over Sample Spaces of Paired Comparisons. Biometrics, 21, 986-1000.
[9] Singh, J. (1976). A Note on Pair Comparison Rankings. The Annals of Statistics, 4(3), 651-654.
[10] Lancaster, J. F., \& Quade, D. (1983). Random Effect in the Paired Comparison Experiments Using the Bradley-Terrey Model. Biometrics, 39, 245-249.
[11] Dittrich, R., Hatzinger, R., \& Katzenbeisser, W. (1998). Modeling the Effect of Subject-Specific Covariates in Paired Comparison Studies with an Application to University Ranking. Journal of the Royal Statistical Society: Series C (Applied Statistics), 47, 511-525.
[12] Glickman, M. E., (1999). Parameters Estimation in Large Dynamic Paired Comparison Experiment. Applied Statistics, 48, 377-394.
[13] Abbas, N., \& Aslam, M. (2009). Prioritizing the Items through Paired Comparison Models. A Bayesian Approach. Pakistan Journal of Statistics, 25, 59-69.
[14] Davidson, R. R., \& Solomon, D. L. (1973). A Bayesian Approach to Paired Comparison Experiments. Biometrika, 60(3), 477-487.
[15] Leonard, T. (1977). An Alternative Approach to the Bradley-Terry Model for Paired Comparisons. Biometrics, 33, 121-132.
[16] Aslam, M. (2002). Reference Prior For the Parameters of Rao-Kupper Model. Proc Pakistan Acad Sci., 39(2), 215-223.
[17] Aslam, M. (2005). Bayesian Comparison of the Paired Comparison Models Allowing Ties. Journal of Statistical Theory and Applications, 4(2), 161-171.
[18] Altaf, S., Aslam, M., \& Aslam, M. (2012). Paired comparison analysis of the van Baaren model using Bayesian approach with noninformative prior. Pakistan Journal of Statistics and Operation Research, 8(2), 259-270.
[19] Altaf, S., Aslam, M., \& Aslam, M. (2013). Bayesian analysis of the van Baaren model for paired comparison. Hacettepe Journal of Mathematics and Statistics, 42(5), 569-80.
[20] Satti, S., \& Aslam, M. (2011). A Bayesian Look at the Pair Comparison Model with Tie and Order Effect. Proc. 8th International Conference on Recent Advances in Statistics, 223-234.
[21] Bernardo, J. M. (1979). Reference Posterior Distribution for Bayesian Inference (with discussion). Journal of statistical planning and inference, 15, 265-278.


[^0]:    $n_{\mathrm{ij}}$ is the total number comparisons made among the two treatments.

