Application of Group Analysis to Find the Function Riemann

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Abstract

This paper is of a synthetic nature, being a result of combining Riemann's method for integrating second-order linear hyperbolic equations with Lie's classification of such equations In paper for the hyperbolic equation was constructed the four-parameter group and with the help of the group was found the solution of the Cauchy problem by the Riemann method for a hyperbolic equation.

1. Introduction

The general solution of the Cauchy problem for a second-order linear hyperbolic partial differential equation in two variables is often given in terms of integrals involving an auxiliary solution called the Riemann function.

By either method, completion of the problem requires the determination of auxiliary function, which is a difficult task because there is no unified method for actually finding this function. Methods for finding Riemann functions have been given by Riemann [1], Copson [4], and Mackie [8]. Regrettably, there are still only a few equations with known Riemann functions.

This paper is of a synthetic nature, being a result of combining Riemann's method [3] for integrating second-order linear hyperbolic equations with Lie's classification [11] of such equations. Using the results for the group classification of homogeneous hyperbolic equation of the second order, it was suggested to find a function of Riemann using the symmetries of the equation.

2. Preliminaries

Let's consider the following hyperbolic equation of the second order:

$$Lv = v_{xx} - v_{yy} + \frac{\mu^2(1-\lambda)\lambda v}{4sh^2\mu(2x+\sigma)} = 0 \left(0 < \lambda < 1, \mu, \sigma \in R\right) \quad (1)$$

in an open domain D, which is bounded by curves of AC $\left(y+x=0\right)$, CB $\left(y-x=0\right)$ and with the section AB $\left(x=-\sigma\right)$.

Let's pose the problem of Cauchy: Find in the domain D function $v(x,y)$, satisfying the conditions

$$v(x,y) \in C(\overline{D}) \cap C^1(D \cup AB) \cup C^2(D), \quad (2)$$

$$Lv(x,y) \equiv 0, \quad (x,y) \in D, \quad (3)$$

$$\lim_{x \to -\sigma}(e^{2\mu(x+y+\sigma)} - e^{-2\mu(x-y+\sigma)})^{\lambda-1} v(x,y) = \tau(y) \quad (4)$$

$$\lim_{x \to -\sigma}(e^{2\mu(x+y+\sigma)} - e^{-2\mu(x-y+\sigma)})^{\lambda} \frac{\partial v}{\partial x} = \nu(y). \quad (5)$$

where $\tau(y)$, $\nu(y)$ − given sufficiently smooth functions.

Applying real coordinate transformation $\xi = x+y$, $\eta = x-y$ equation (1) leads to the canonical form:

$$u_{\xi\eta} + \frac{\mu^2(1-\lambda)\lambda u}{sh^2\mu(r+\sigma)} = 0, \quad (6)$$

where $r = \xi + \eta$ and boundary values are of the form

$$\lim_{\xi+\eta\to-2\sigma}(e^{2\mu(\xi+\sigma)} - e^{-2\mu(\eta+\sigma)})^{\lambda-1}u(\xi,\eta) = \tilde{\tau}(\xi) ,$$

$$\lim_{\xi+\eta\to-2\sigma}(e^{2\mu(\xi+\sigma)} - e^{-2\mu(\eta+\sigma)})^{\lambda}\left(\frac{\partial u(\xi,\eta)}{\partial \eta} + \frac{\partial u(\xi,\eta)}{\partial \xi}\right) = \tilde{v}(\xi)$$

We shall define the operator L by the identity

$$Lu = u_{\xi\eta} + a(\xi,\eta)u_{\xi} + b(\xi,\eta)u_{\eta} + c(\xi,\eta)u , \qquad (7)$$

The operator L^* defined by the identity

$$L^*v = v_{\xi\eta} - (av)_{\xi} - (bv)_{\eta} + cv$$

is known as the adjoint of L.

To solve the problem we use the method of Riemann, which is based on the following identity:

$$2(vLu - uL^*v) = (vu_{\eta} - uv_{\eta} + 2auv)_{\xi} + (vu_{\xi} - uv_{\xi} + 2buv)_{\eta} \quad (8)$$

If the functions v and u are such that $Lu = Lv = 0$ throughout a domain G bounded by sufficiently smooth closed curve Γ. Then an application of Green's theorem yields

$$2\iint_{G}(vLu - uL^*v)d\xi d\eta =$$
$$= \oint_{\Gamma}[-(vu_{\xi} - uv_{\xi} + 2buv)d\xi + (vu_{\eta} - uv_{\eta} + 2auv)d\eta]$$

Riemann's method reduces the problem of integration of the equation

$$Lu = f$$

To that of constructing the auxiliary function $v = R(\xi,\eta;\xi_0,\eta_0)$ solving the adjoint equation:

$$L^*R = 0$$

and satisfying the following conditions on the characteristics:

$$(R_{\eta} - aR)\big|_{\xi=\xi_0} = 0,$$
$$(R_{\xi} - bR)\big|_{\eta=\eta_0} = 0, \qquad (9)$$
$$R(\xi_0,\eta_0;\xi_0,\eta_0) = 1.$$

Provided that the function v is known, the solution of the Cauchy problem:

$$Lu = f , \; u\big|_{\Gamma} = u_0(x), \; \frac{\partial u}{\partial n}\bigg|_{\Gamma} = u_1(x) \qquad (10)$$

with data on an arbitrary non-characteristic curve Γ is given by the formula

$$u(\xi_0,\eta_0) = \frac{(uR)_P + (uR)_Q}{2} - \frac{1}{2}\int_{PQ} -(Ru_{\xi} - uR_{\xi} + 2buR)d\xi$$
$$+ (Ru_{\eta} - uR_{\eta} + 2auR)d\eta + \iint_{G}Rfd\xi d\eta \qquad , \quad (11)$$

where the double integral is taken over the domain bounded by the characteristics $\xi = \xi_0$, $\eta = \eta_0$ and the curve Γ. The function $v = R(\xi,\eta;\xi_0,\eta_0)$ is called Riemann's function, and the boundary-value problem (10) is called the characteristic Cauchy problem, or the Goursat problem. The solution of the Goursat problem is unique.

3. Main Results

In our case, the equation adjoint equation has the form

$$u_{\xi\eta} + \frac{\mu^2(1-\lambda)\lambda}{sh^2\mu(r+\sigma)}u = 0 \qquad (12)$$

Let's note that in our case the desired function of Riemann $v = R(\xi,\eta;\xi_0,\eta_0)$ satisfies the following conditions on the characteristics:

$$R\big|_{\eta=\eta_0} = R\big|_{\xi=\xi_0} = 1, \quad R(\xi_0,\eta_0;\xi_0,\eta_0) = 1. \qquad (13)$$

The symmetry operator of the homogeneous equation $Lu = 0$ has the form [6]:

$$X = v(\xi)\frac{\partial}{\partial \xi} + w(\eta)\frac{\partial}{\partial \eta} + \tau(\xi,\eta)u\frac{\partial}{\partial u} .$$

Thus, as follows from [2], must be done the following relations:

$$\frac{\partial \tau}{\partial \xi} + \frac{\partial(bv)}{\partial \xi} + w\frac{\partial b}{\partial \eta} = 0, \qquad \frac{\partial \tau}{\partial \eta} + \frac{\partial(aw)}{\partial \eta} + v\frac{\partial a}{\partial \xi} = 0,$$
$$\frac{\partial^2 \tau}{\partial \xi \partial \eta} + a\frac{\partial \tau}{\partial \xi} + b\frac{\partial \tau}{\partial \eta} + \frac{\partial(cv)}{\partial \xi} + \frac{\partial(cw)}{\partial \eta} = 0.$$

Substituting in this case $a = 0$, $b = 0$, $c = \dfrac{\lambda}{sh^2\mu(r+\sigma)}$, we'll obtain the following relation

$$v_{\xi}sh\mu(r+\sigma) - 2v\mu ch\mu(r+\sigma)$$
$$= -w_{\eta}sh\mu(r+\sigma) + 2w\mu ch\mu(r+\sigma).$$

The solution of this partial differential equation of the first order will functions

$$v = Ae^{2\mu(\xi+\sigma)} + Be^{-2\mu(\xi+\sigma)} + C ,$$

$$w = -Ae^{-2\mu(\eta+\sigma)} - Be^{2\mu(\eta+\sigma)} - C , \; \tau = D .$$

where A, B, C, D – arbitrary constants. Thus, equation (12) admits the three-parameter group (in addition to stretching of the variable u and the infinite group consisting of addition to u of any solution of the equation; this group is common to all

linear equations) with the generators

$$X_1 = e^{2\mu(\xi+\sigma)}\frac{\partial}{\partial\xi} - e^{-2\mu(\eta+\sigma)}\frac{\partial}{\partial\eta},$$

$$X_2 = e^{-2\mu(\xi+\sigma)}\frac{\partial}{\partial\xi} - e^{2\mu(\eta+\sigma)}\frac{\partial}{\partial\eta},$$

$$X_3 = \frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}.$$

Let us find a linear combination of these operators,

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ − arbitrary constants.

Following [7], let us require first the invariance of the characteristics $\xi = \xi_0$, $\eta = \eta_0$. The invariance test has the form:

$$X(\xi - \xi_0) = 0, \ \ X(\eta - \eta_0) = 0.$$

One can set $\alpha_3 = 1$, then obtain $\alpha_2 = e^{2\mu(\xi_0+\sigma)}$, $\alpha_1 = e^{2\mu(\eta_0+\sigma)}$. One can readily verify that the resulting operator

$$X = \frac{\left(e^{2\mu(\xi+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)\left(e^{2\mu(\xi+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)}{e^{2\mu(\xi+\sigma)}e^{-2\mu(\eta_0+\sigma)}}\frac{\partial}{\partial\xi}$$

$$- \frac{\left(e^{2\mu(\eta+\sigma)} - e^{2\mu(\eta_0+\sigma)}\right)\left(e^{2\mu(\eta+\sigma)} - e^{-2\mu(\xi_0+\sigma)}\right)}{e^{2\mu(\eta+\sigma)}e^{-2\mu(\xi_0+\sigma)}}\frac{\partial}{\partial\eta}$$

is admitted by the Goursat problem (9).

Invariant of this operator have the form

$$I = \frac{\left(e^{2\mu(\xi+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)\left(e^{-2\mu(\eta+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)}{\left(e^{-2\mu(\eta+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)\left(e^{2\mu(\xi+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)}.$$

Now if f is assumed to be a function of

$$z = \frac{\left(e^{2\mu(\xi+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)\left(e^{-2\mu(\eta+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)}{\left(e^{-2\mu(\eta+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)\left(e^{2\mu(\xi+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)}$$

the equation (12) becomes

$$z(1-z)^2 f''(z) + (1-z)^2 f'(z) + (1-\lambda)\lambda f(z) = 0,$$

With the change of variable $z = \dfrac{\mu}{\mu-1}$, obtain

$$\mu(1-\mu)f''(\mu) + (1-2\mu)f'(\mu) - (1-\lambda)\lambda f(\mu) = 0$$

The solution of the obtained equation is function Gauss

$$f = F(\lambda, 1-\lambda, 1; \mu).$$

Then the Riemann's function in the (ξ, η) will have the

form

$$R(\xi, \eta, \xi_0, \eta_0) =$$

$$F\left(\lambda, 1-\lambda, 1; \frac{\left(e^{2\mu(\xi+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)\left(e^{-2\mu(\eta+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)}{\left(e^{2\mu(\xi_0+\sigma)} - e^{-2\mu(\eta_0+\sigma)}\right)\left(e^{2\mu(\xi+\sigma)} - e^{-2\mu(\eta+\sigma)}\right)}\right)$$

Substituting in the formula $a = 0$, $b = 0$, $f = 0$ obtain

$$u(\xi_0, \eta_0) = \frac{2\beta\mu\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\int_{-\eta_0-2\sigma}^{\xi_0}\frac{\left(e^{-2\mu(\eta_0+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)^\lambda e^{2\mu(t+\sigma)}\tilde{\tau}(t)dt}{\left(e^{-2\mu(\eta_0+\sigma)} - e^{2\mu(t+\sigma)}\right)^\lambda\left(e^{2\mu(t+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)^\lambda}$$

$$+ \frac{2\beta\mu\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\int_{-\eta_0-2\sigma}^{\xi_0}\frac{\left(e^{-2\mu(\eta_0+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)^\lambda \tilde{\nu}(t)dt}{\left(e^{-2\mu(\eta_0+\sigma)} - e^{2\mu(t+\sigma)}\right)^\lambda\left(e^{2\mu(t+\sigma)} - e^{2\mu(\xi_0+\sigma)}\right)^\lambda}$$

Returning to the old variables x and y, we'll get the solution of the Cauchy's problem

$$u(x, y) = \int_{y-x-2\sigma}^{x+y}\frac{\left(e^{-2\mu(x-y+\sigma)} - e^{2\mu(x+y+\sigma)}\right)^\lambda e^{2\mu(t+\sigma)}\tau(t)dt}{\left(e^{-2\mu(x-y+\sigma)} - e^{2\mu(t+\sigma)}\right)^\lambda\left(e^{2\mu(t+\sigma)} - e^{2\mu(x+y+\sigma)}\right)^\lambda}$$

$$+ \int_{-y-x-2\sigma}^{x+y}\frac{\left(e^{-2\mu(x-y+\sigma)} - e^{2\mu(x+y+\sigma)}\right)^\lambda \nu(t)dt}{\left(e^{-2\mu(x-y+\sigma)} - e^{2\mu(t+\sigma)}\right)^\lambda\left(e^{2\mu(t+\sigma)} - e^{2\mu(x+y+\sigma)}\right)^\lambda} \quad (14)$$

Theorem. If functions $\tau(y) \in C^2[-\delta; \delta]$, $\nu(y) \in C^2[-\delta; \delta]$, then the Cauchy's problem for equation (1) has a unique solution, which is defined by (14).

4. Conclusion

The crux of Riemann's method is the replacement of the original initial value problem by a standard problem which depends only on the coefficients in equation. Once obtained, Riemann's function, of this standard problem provides us with the solution (11) to every Caushy problem of any initial values. The group-theoretic approach presented above provides one of the most effective ways of finding the Riemann function.

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