# Open Science Journal of Mathematics and Application 

2015; 3(5): 147-151

Published online September 22, 2015 (http://www.openscienceonline.com/journal/osjma)

# Properties of Operator Systems in Hilbert Spaces 

N. B. Okelo<br>School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, Bondo, Kenya<br>\section*{Email address}<br>bnyaare@yahoo.com

## To cite this article

N. B. Okelo. Properties of Operator Systems in Hilbert Spaces. Open Science Journal of Mathematics and Application. Vol. 3, No. 5, 2015, pp. 147-151.


#### Abstract

Tensor products, operator systems and spectral theory of operators form a very important focal point in functional analysis. The objective of this study has been to characterize the properties of operator systems and subsystems in Hilbert spaces. The methodology involved the use of tensor products, eigenvalues and eigenvectors. The results obtained show that operator systems in Hilbert spaces can be divided into subsystems without altering their structures. The results are significant in applications in quantum mechanics.


## Keywords

Resultant, Operator, Multiparameter System, Eigenvalue, Eigenvectors and Tensor Products

## 1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The development of physical sciences is becoming more and more a challenge to mathematicians. In particular, the resolution of the problems associated with the physical processes and, consequently, the study of partial differential equations and mathematical physics equations, requires a new approach. The method of separation of variables in many cases turns out to be the only acceptable, since it reduces finding a solution to a complex equation with many variables to find a solution to a system of ordinary differential equations, which are much easier to study.

## 2. Preliminaries

We give some definitions and concepts from the theory of multiparameter operator systems necessary for understanding of the further considerations.

Let the linear multiparameter system be in the form:

$$
\begin{align*}
& B_{k}(\lambda) x_{k}=\left(B_{0, k}+\sum_{i=1}^{n} \lambda_{i} B_{i, k}\right) x_{k}=0,  \tag{1}\\
& k=1,2, \ldots, n
\end{align*}
$$

where operators $B_{k, i}$ act in the Hilbert space $H_{i}$

Definition 1. $[1,2,11] \quad \lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right) \in C^{n}$ is an eigenvalue of the system (1) if there are non-zero elements $x_{i} \in H_{i}, i=1,2, \ldots, n$ such that (1) is satisfied, and decomposable tensor $x=x_{1} \otimes x_{2} \otimes \ldots \otimes x_{n}$ is called the eigenvector corresponding to eigenvalue $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right) \in C^{n}$.

Definition 2. The operator $B_{s, i}^{+}$is induced by an operator $B_{s, i}$, acting in the space $H_{i}$, into the tensor space $H=H_{1} \otimes \ldots \otimes H_{n}$, if on each decomposable tensor $x=x_{1} \otimes \ldots \otimes x_{n}$ of tensor product space $H=H_{1} \otimes \ldots \otimes H_{n}$ we have $B_{s, i}^{+} x=x_{1} \otimes \ldots \otimes x_{i-1} \otimes B_{s, i} x_{i} \otimes x_{i+1} \otimes \ldots \otimes x_{n}$ and on all the other elements of $H=H_{1} \otimes \ldots \otimes H_{n}$ the operator $B_{s, i}^{+}$is defined on linearity and continuity.
Definition 3 ([5], [6]).
Let $\quad x_{0, \ldots, 0}=x_{1} \otimes x_{2} \otimes \ldots \otimes x_{n}$ be an eigenvector of the system (1), corresponding to its eigenvalue $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$; then $x_{m_{1} \ldots, m_{n}}$ is $m_{1}, m_{2}, \ldots, m_{n}$ - th associated vector (see [4]) to an eigenvector $x_{0,0, \ldots, 0}$ of the system (1) if there is a set of vectors $\left\{x_{i_{1}, i_{2}, \ldots, i_{n}}\right\} \subset H_{1} \otimes \cdots \otimes H_{n}$, satisfying to conditions

$$
\begin{gather*}
B_{0, i}^{+}(\lambda) x_{i s, s_{2}, \ldots, s_{n}}+B_{1, i}^{+} x_{s_{1}-1, s_{2}, \ldots, s_{n}}+\ldots+B_{n, i}^{+} x_{s_{1}, \ldots, s_{n-1}, s_{n}-1}=0 . \\
x_{i_{1}, 1, s_{2}, \ldots, s_{n}}=0, \text { when } s_{i}<0 \tag{2}
\end{gather*}
$$

$$
0 \leq s_{r} \leq m_{r}, r=1,2, \ldots, n, i=1, \ldots, n
$$

For the indices $s_{1}, s_{2}, \ldots, s_{n}$ in element $\left(x_{i_{1}, i_{2}, \ldots, i_{n}}\right) \subset H_{1} \otimes \cdots \otimes H_{n}$, there are various arrangements from set of integers on $n$ with $0 \leq s_{r} \leq m_{r}, r=1,2, \ldots, n$, .

Definition 4. In $[1,3,11]$ the system (1) is an analogue of the Cramer's determinants, when the number of equations is equal to the number of variables, and is defined as follows: On decomposable tensor $x=x_{1} \otimes \ldots \otimes x_{n} \quad$ operators $\Delta_{i} \quad$ are defined with help the matrices

$$
\sum_{i=0}^{n} \alpha_{i} \Delta_{i} x=\otimes \otimes\left(\begin{array}{ccccc}
\alpha_{0} & \alpha_{1} & \alpha_{2} & \ldots & \alpha_{n}  \tag{3}\\
B_{0,1} x_{1} & B_{1,1} x_{1} & B_{2,1} x_{1} & \ldots & B_{n, 1} x_{1} \\
B_{0,2} x_{2} & B_{1,2} x_{2} & B_{2,2} x_{2} & \ldots & B_{n, 2} x_{2} \\
B_{0,3} x_{3} & B_{1,3} x_{3} & B_{2,3} x_{3} & \ldots & B_{n, 3} x_{3} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
B_{0, n} x_{n} & B_{1, n} x_{n} & B_{2, n} & \ldots & B_{n, n}
\end{array}\right)
$$

where $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{n}$ are arbitrary complex numbers, under the expansion of the determinant means its formal expansion, when the element $x=x_{1} \otimes x_{2} \otimes \ldots \otimes x_{n}$ is the tensor products of elements $x_{1}, x_{2}, \ldots, x_{n}$ If $\alpha_{k}=1, \alpha_{i}=0, i \neq k$, ,then right side of (10) equal to $\Delta_{k} x$, where $x=x_{1} \otimes x_{2} \otimes \ldots \otimes x_{n}$ On all the other elements of the space $H$ operators $\Delta_{i}$ are defined by linearity and continuity. $E_{s}(s=1,2, \ldots, n)$ is the identity operator of the space $H_{i}$. Suppose that for all $x \neq 0$, $\left(\Delta_{0} x, x\right) \geq \delta(x, x), \delta>0$, and all $B_{i, k}$ are selfadjoint operators in the space $H_{i}$. Inner product [...] is defined as follows; if $x=x_{1} \otimes x_{2} \otimes \ldots \otimes x_{n}$ and $y=y_{1} \otimes y_{2} \otimes \ldots \otimes y_{n}$ are decomposable tensors, then $[x, y]=\left(\Delta_{0} x, y\right)$ where $\left(x_{i}, y_{i}\right)$ is the inner product in the space. $H_{i}$. On all the other elements of the space $H$ the inner product is defined on linearity and continuity. In space $H$ with such a metric all operators $\Gamma_{i}=\Delta_{0}^{-1} \Delta_{i}$ are selfadjoin

Definition 5. ([7],[8] [10])
Let two operator pencils depending on the same parameter andacting in, generally speaking, in various Hilbert spaces be as follows

$$
\begin{aligned}
& A(\lambda)=A_{0}+\lambda A_{1}+\lambda^{2} A_{2}+\ldots+\lambda^{n} A_{n}, \\
& B(\lambda)=B_{0}+\lambda B_{1}+\lambda^{2} B_{2}+\ldots+\lambda^{m} B_{m}
\end{aligned}
$$

Operator $\operatorname{Re} s(A(\lambda), B(\lambda))$ is presented by the matrix

$$
\left(\begin{array}{cccccc}
A_{0} \otimes E_{2} & A_{1} \otimes E_{2} & \ldots & A_{n} \otimes E_{2} & \ldots & 0 \\
\cdot & \cdot & \ldots & \cdot & \ldots & . \\
0 & 0 & \ldots A_{0} \otimes E_{2} & A_{1} \otimes E_{2} & \ldots & A_{n} \otimes E_{2} \\
E_{1} \otimes B_{0} & E_{1} \otimes B_{1} & \ldots & E_{1} \otimes B_{m} & . . & 0 \\
\cdot & \cdot & \ldots & \cdot & \ldots & . \\
\cdot & \cdot & \ldots E_{1} \otimes B_{0} & E_{1} \otimes B_{1} & \ldots & E_{1} \otimes B_{m}
\end{array}\right)
$$

whichacts in the $\left(H_{1} \otimes H_{2}\right)^{n+m}$ - direct sum of $n+m$ copiesof the space $H_{1} \otimes H_{2}$ In a matrix (4), the number of rows with operators $A_{i}$ is equal to leading degree of the parameter $\lambda$ in pencils $B(\lambda)$ and the number of rows with $B_{i}$ is equal to the leading degree of parameter $\lambda$ in $A(\lambda)$. The notion of abstract analog of resultant of two operator pencils is considered in the [7] for the case of the same leading degree of the parameter in both pencils and in the [2] for, generally speaking, different degree of the parameters in the operator pencils.

## Theorem 1 [7, 8].

Let for all operators bounded in corresponding Hilbert spaces, one of operators $A_{n}$ or $B_{m}$ has bounded inverse. Then operator pencils $A(\lambda)$ and $B(\lambda)$ have a common point of spectra if and only if

$$
\operatorname{Ker} \operatorname{Re} s(A(\lambda), B(\lambda)) \neq\{\vartheta\}
$$

Remarkl. If the Hilbert spaces $H_{1}$ and $H_{2}$ are the finite dimensional spaces then a common points of spectra of operator pencils $A(\lambda)$ and $B(\lambda)$ are their common eigenvalues. (see [6], [7].)

$$
\left\{B_{i}(\lambda)=B_{0, i}+\lambda B_{1, i}+\ldots+\lambda^{k_{i}} B_{k_{i}, i}, \quad i=1,2, \ldots, n\right.
$$

$B_{i}(\lambda)$ - operator bundles acting in a finite dimensional Hilbert space $H_{i}$ correspondingly. Suppose that $k_{1} \geq k_{2} \geq \ldots \geq k_{n}$. In the space $H^{k_{1}+k_{2}}$ (the direct sum of $k_{1}+k_{2}$ tensor product $H=H_{1} \otimes \ldots \otimes H_{n} \quad$ of $\quad$ spaces $\left.H_{1}, H_{2}, \ldots, H_{n}\right)$ are introduced the operators $R_{i}(i=1, \ldots, n-1)$ with the help of operational matrices (3.12) Let $B_{i}(\lambda)$ be the operational bundles acting in a finite dimensional Hilbert space $H_{i}$, correspondingly. Without loss ofcopies with

$$
R_{i-1}=\left(\begin{array}{cccccc}
B_{0,1}^{+} & B_{1,1}^{+} & \cdots & B_{k_{1}, 1}^{+} & \ldots & 0 \\
0 & B_{0,1}^{+} & B_{1,1}^{+} \cdot \cdot & B_{k_{1}-1,1}^{+} & B_{k_{1}, 1}^{+} \cdot \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdot B_{0,1}^{+} & B_{1,1}^{+} & \cdots & B_{k_{1}, 1}^{+} \\
B_{0, i}^{+} & B_{1, i}^{+} & \cdots & B_{k_{i}, i}^{+} & 0 . \cdot & 0 \\
0 & B_{0, i}^{+} & B_{1, i}^{+} \cdots & \cdot & B_{k_{i}, i}^{+} \cdots & 0 \\
\cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\
0 & 0 & \cdots B_{0, i}^{+} & B_{1, i}^{+} & \cdots & B_{k_{i}, i}^{+}
\end{array}\right),
$$

The number of rows with operators $B_{s, 1}, s=0,1, \ldots, k_{1}$ in the matrix $R_{i-1}$ is equal to $k_{2}$ and the number of rows with operators $B_{s, i}, s=0,1, \ldots, k_{i}$ is equal to $k_{1}$. We designate $\sigma_{p}\left(B_{i}(\lambda)\right)$ the set of eigenvalues of an operator $B_{i}(\lambda)$. From [5] we have the result:

Theorem 2. [9] $\bigcap_{i=1}^{n} \sigma_{p}\left(B_{i}(\lambda)\right) \neq\{\theta\} \quad$ if and only if $\bigcap_{i=1}^{n-1} \operatorname{Ker} R_{i} \neq\{\theta\},\left(\operatorname{Ker} B_{k_{1}}=\{\theta\}\right)$.

## 3. Main Results

Consider the system

$$
\begin{align*}
A_{i, j, s}(\lambda) x_{s} & =\left(A_{0, s}+\sum_{r=1}^{k_{1, s}} \lambda_{1}^{r} A_{1, r, s}+\ldots+\sum_{r=1}^{k_{n, s}} \lambda_{n}^{r} A_{n, r, s}+\right. \\
& +\sum \lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}} A_{i_{1}, \ldots, i_{n}}  \tag{4}\\
& s=1,2, \ldots, n
\end{align*}
$$

The parameters $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ enter the system nonlinearly, and the system (4) contains also the products of these parameters. Divide the system of equations (4) into groups of $n$ in each group. If some equations remains outside, these equations we add by others operators from the system (4). Each group contains $n$ operators and will be considered separately.

In (4) the coefficients of the parameter $\lambda_{m}^{r}, r \leq k_{m}, m=1,2, \ldots, n$ are the operators $A_{i, m, j}$, which act in the space $H_{j}$, index $i$ indicate on the parameter $\lambda_{i}$, index $k$ - on the degree of the parameter $\lambda_{i}$.

We introduce the notations:

$$
\begin{equation*}
\lambda_{m}^{r}=\lambda_{k_{1}+k_{2}+\ldots+k_{m-1}+r}, r \leq k_{m}, \quad m=1,2, \ldots, n \tag{5}
\end{equation*}
$$

Further, we numerate the different products of variables $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ in the system (4) on increasing of the degrees of the parameter $\lambda_{1}$. Let the numbers of term with the products of the parameters $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ are equal to $r$ Put further

$$
\lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}}=\left(\lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}}\right)_{t}=\tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{n}+t}, t \leq r,
$$

where $t \leq s$ is the number which correspond the multiplier at $\lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}}$ the ordering of multiplies of parameters in the system (4). So in new notations to the product $\lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}}$ correspond the parameter $\tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{n}+t}, t \leq r \quad\left(\lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}} \rightarrow\right.$ $\left.\tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{n}+t}, t \leq r\right)$, accordingly, operators

$$
\begin{gather*}
A_{r, s, i}=D_{k_{1}+k_{2}+\ldots+k_{s-1}+s, i}, r=1,2, \ldots, n ; s=1,2, \ldots, k_{r} \\
i=1,2, \ldots, n \\
k_{r}=\max k_{r, i}, i=1,2, \ldots, k  \tag{6}\\
A_{k_{1}, k_{2} \ldots, k_{n} ; i}=D_{k_{1}+k_{2}+\ldots+k_{m}+t, i}, t=1,2, \ldots, s ; ; i=1,2, \ldots, n
\end{gather*}
$$

When $s$ is the number of different products of parameters, entering the system(4).

In new notations the system (4) in the tensor product of
spaces $\quad H_{1} \otimes H_{2} \otimes \ldots \otimes H_{n} \quad$ contains $\quad k_{1}+k_{2}+\ldots+k_{n}+s$ parameters and $n$ equations.Let $k_{1}+k_{2}+\ldots+k_{n}=k$ Then

$$
\begin{align*}
& \sum_{r=0}^{n} \sum_{k=1}^{k_{r}}\left[\tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{r-1}+k} D_{k_{1}+k_{2}+\ldots+k_{r-1}+k, i}\right] x_{i}+\left[\sum_{k=1}^{r} \tilde{\lambda}_{k+1} D_{k+t, i}\right] x_{i=0}=0  \tag{7}\\
& k_{0}=0 ; \quad k_{-i}=0 ; i=1,2, \ldots, n
\end{align*}
$$

Adding the system (7) with help of new equations so manner that the connections between the parameters, following from the equations of the system (4), satisfy. Introduce the operators $T_{0}, T_{1}, T_{2}, \bar{T}_{0}, \bar{T}_{0}$ acting in the finite dimensional space $R^{2}$ and defining with help of the matrices

$$
\begin{align*}
& T_{0}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad T_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right), T_{2}=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right), \overline{\bar{T}}_{0}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right), \\
& T_{1, s_{s}, r}=\left(\begin{array}{lllllll}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{array}\right), \ldots T_{k_{n}+1, s_{n}, r}=\left(\begin{array}{lllllll}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{array}\right) \\
& T_{\left(s_{1}, s_{2}, \ldots, s_{n}\right)_{r}}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0
\end{array}\right) \tag{8}
\end{align*}
$$

The number 1 stands on the diagonal elements of the first $s_{1}$ rows of the matrix $T_{1, s_{1}, r}$; diagonal elements of the rows $s_{1}+s_{2}+\ldots+s_{i-1}+1, \ldots, s_{1}+s_{2}+\ldots+s_{i}$ of the matrix $T_{k_{i}+1, s_{i+1}, r}$ is equal also to 1 and so on. Besides, all matrices $T_{1, s_{1}, r}, \ldots, T_{k_{i}+1, s_{i+1}, r}, \ldots, T_{\left(s_{1}, s_{2}, \ldots, s_{n}\right)_{r}}$ have the order $s_{1}+s_{2}+\ldots+s_{n}$.

Adding the system (7) by the following equations

$$
\left(T_{2, n+1}+\tilde{\lambda}_{1} T_{0, n+1}+\tilde{\lambda}_{2} T_{1, n+1}\right) x_{n+1}=0
$$

$\left(\tilde{\lambda}_{k_{1}+k_{2}-2} T_{2, n+k_{1}+k_{2}-2}+\tilde{\lambda}_{k_{1}+k_{2}-1} T_{0, n+k_{1}+k_{2}-2}+\right.$
$\left.+\tilde{\lambda}_{k_{1}+k_{2}} T_{1, n+k_{1}+k_{2}-2}\right) x_{n+k_{1}+k_{2}-2}=0$

$$
\begin{gather*}
\left(\tilde{\lambda}_{k_{1}+\ldots+k_{n-1}-2} T_{2,1+\sum_{i=1}^{n-1} k_{i}}+\tilde{\lambda}_{k_{1}+\ldots+k_{n-1}-1,0} T_{0,1+\sum_{i=1}^{n-1} k_{i}}^{n+\ldots}\right) x_{n+k_{1}+\ldots+k_{n-1}-2}=0 \\
\left(\tilde{\lambda}_{k_{1}+\ldots+k_{n}-2} T_{2}+\tilde{\lambda}_{k_{1}+\ldots+k_{n}-1} T_{0}+\tilde{\lambda}_{k} T_{1}\right) x_{k}=0  \tag{9}\\
x_{s} \in R^{2}, s>n
\end{gather*}
$$

$$
\begin{aligned}
& \left(T_{0, t}+\tilde{\lambda}_{1} T_{i_{1}, t}+\tilde{\lambda}_{k_{1}+1} T_{i_{2}, t}+\ldots+\tilde{\lambda}_{k_{1}+\ldots+k_{n-1}+1} T_{i_{n}, t}-\right. \\
& \left.-\tilde{\lambda}_{k+\left(i_{1}, i_{2}, \ldots, i_{n}\right) t} T_{\left(i_{1}, i_{2}, \ldots, i_{n}\right)}\right) x_{t}=0 \\
& t=1,2, \ldots, r
\end{aligned}
$$

Denote $\tilde{\lambda}_{k+\left(i_{1}, i_{2}, \ldots, i_{n}\right)_{r}}$ the multiplier $\lambda_{1}^{i_{1}} \lambda_{2}^{i_{2}} \ldots \lambda_{n}^{i_{n}}$ of the parameters, entering the system (4) having the coefficient $A_{\left(i_{1}, i_{2}, \ldots, i_{n}\right)_{r}}$. System ((4), (9)) form the linear multiparameter system, containing $k_{1}+k_{2}+\ldots+k_{n}+r$ equations and $k_{1}+k_{2}+\ldots+k_{n}+s$ parameters. To this system we may apply all results, given in the beginning of this paper.

Theorem3. [4]. Let the following conditions:
a) operators $A_{k, t}, A_{k_{1}, k_{2}, \ldots, k_{n} ; t}$ in the space $H_{i}$ are bounded at the all meanings $i$ and $k$.
b) operator $\Delta_{0}^{-1}$ exists and boundedsatisfy:

Then the system of eigen and associated vectors of (4) coincides with the system of eigen and associated vectors of each operators $\quad \Gamma_{i}(i=1,2, \ldots, n)$

Given two equations from (9). Let the equations be:

$$
\begin{align*}
& \left(T_{2}+\lambda_{1} T_{0}+\lambda_{2} T_{1}\right) x_{n+1}=0 \\
& \left(\lambda_{1} T_{2}+\lambda_{2} T_{0}+\lambda_{3} T_{1}\right) x_{n+2}=0 \tag{10}
\end{align*}
$$

Let $\lambda_{1} \neq 0$ и $x_{n+1}=\left(\alpha_{1}, \beta_{1}\right) \neq 0$ is the component of the eigenvector of the system ((4),(9)). We have

$$
\begin{aligned}
& \left(\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)+\lambda_{1}\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)+\lambda_{2}\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)\right)\left(\alpha_{1}, \beta_{1}\right)=0, \\
& \lambda_{1} \beta_{1}+\lambda_{2} \alpha_{1}=0, \beta_{1}+\lambda_{1} \alpha_{1}=0, \lambda_{2} \neq 0 ; \lambda_{2}=\lambda_{1}^{2} .
\end{aligned}
$$

Further from the condition $\lambda_{1} \neq 0, \lambda_{2} \neq 0, x_{n+2}=\left(\alpha_{2}, \beta_{2}\right) \neq 0 \quad$ it follows $\lambda_{2} \beta_{2}+\lambda_{3} \alpha_{2}=0$, $\lambda_{1} \beta_{2}+\lambda_{2} \alpha_{1}=0$ and consequently, $\tilde{\lambda}_{1} \tilde{\lambda}_{3}=\tilde{\lambda}_{2}^{2}$. Earlier we proved that $\tilde{\lambda}_{2}=\lambda_{1}^{2}$, Consequently, $\tilde{\lambda}_{3}=\lambda_{1}^{3}$.

On analogy for other parameters of ((4),(9)): if $\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{n}+s}\right)$ is the eigenvalue of the system -((4), (9)), then $\lambda_{4}=\lambda_{1}^{4}, \ldots, \lambda_{k_{1}}=\lambda_{1}^{k_{1}}, \ldots, \tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{r}+s}=\lambda_{r+1}^{s}$, $r=1,2, \ldots, n-1 ; s=1,2, \ldots, k_{n}$.

To each multiplier of parameters $\left(\tilde{\lambda}_{j_{1}}^{r_{1}} \tilde{\lambda}_{j_{2}}^{r_{j_{2}}} \ldots \tilde{\lambda}_{j_{k}}^{r_{j k}}\right)_{t}=\tilde{\lambda}_{k+t} ; t \leq r$ it is corresponded the equation

$$
\begin{aligned}
& \left(T_{0, t+k}+\tilde{\lambda}_{1} T_{1, i_{1}, k+t}+\tilde{\lambda}_{k_{1}+1} T_{2, i_{2}, k+t}+\ldots+\tilde{\lambda}_{k_{1}+k_{2}+\ldots+k_{n-1}+1} T_{n, i_{n}, k+t}-\right. \\
& \left.-\tilde{\lambda}_{k+\left(i_{1}, i_{2}, \ldots, i_{n}\right)_{t}} T_{\left(i_{1}, \ldots, i_{n}\right)_{t}, t+k}\right) x_{k+t}=0
\end{aligned}
$$

Consider the last equation, in which

$$
\left.\begin{array}{l}
T_{1, s_{1}, r}=\left(\begin{array}{ccccccc}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{array}\right), \ldots, \\
T_{k_{1}+\ldots+k_{n-1}+1, s_{n}, r}=\left(\begin{array}{lllllll}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{array}\right) \\
\tilde{T}_{\left(s_{1}, \ldots, s_{n}\right)_{t}, r}=\left(\begin{array}{lllllll}
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{array}\right) \\
\tilde{T}_{0, r}=\left(\begin{array}{llllll} 
\\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
0 \\
. & . & . & . & . & . \\
0 & . \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{array}\right)
\end{array}\right)
$$

For operators, defining with help the matrices $T_{1, s_{1}, k+t}, T_{2, s_{2}, k+t}, \ldots, T_{n, s_{n}, k+t}, T_{0 . k+t}$ act in space $R^{s_{1}+\ldots+s_{n}}$ On eigenvector $\left(\alpha_{1}, \ldots, \alpha_{s_{1}+s_{2}+\ldots+s_{n}}\right) \in R^{s_{1}+\ldots+s_{n}}$.

$$
\begin{aligned}
& \left(-\vec{T}_{0, r}+\tilde{\lambda}_{1} T_{1, s_{1}, r}+\ldots+\tilde{\lambda}_{1+\sum_{i=1}^{n-1} k} T_{1+k_{1}+\ldots+k_{n-1}, s_{n}, r}\right) \tilde{\alpha}= \\
& \left.=\tilde{\lambda}_{k+t} T_{\left(s_{1} \ldots s_{n}\right.} r\right) \tilde{\alpha}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& \tilde{\lambda}_{1} \alpha_{1}=\tilde{\lambda}_{k+1} \alpha_{s_{1}+\ldots+s_{n}} \\
& \tilde{\lambda}_{1} \alpha_{s_{1}}=\alpha_{s_{1}-1} \\
& \tilde{\lambda}_{k_{1}+1} \alpha_{s_{i}+1}=\alpha_{s_{1}} \\
& \tilde{\lambda}_{k_{1}+1} \alpha_{s_{1}+s_{2}}=\alpha_{s_{1}+s_{2}-1}
\end{aligned}
$$

Hence, $\lambda_{1}^{s_{1}} \lambda_{2}^{s_{2}} \cdots \lambda_{n}^{s_{n}}=\lambda_{k+s} ; s \leq r$.
For the obtained linear multiparameter system we construct operator $\Delta_{0}$ on rule (3).

The condition $\operatorname{Ker} \Delta_{0}^{-1}=\{\vartheta\}$ means that operators $\Gamma_{i}=\Delta_{0}^{-1} \Delta_{i}$ are pair commute [2]. So operators $\Gamma_{i}$ act in finite dimensional space $H$ and operators $\Gamma_{k_{1}+k_{2}+\ldots+k_{r-1}+1}$ have not the zero eigenvalues then for the any eigenvalue $\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k_{1}+k_{2}+\ldots+k_{n}}\right)$ of the system((4),(9)) and from Equation 2.47 and Equation 2.48 in [7] it follows that there is such eigen element $z$ that the equalities,
$\Gamma_{i, s} z=\lambda_{i, s} z, i=1,2, \ldots, k_{1}+k_{2}+\ldots+k_{n}$ satisfy. For analogy conditions we obtain the analogy results for all groups. We have the several systems of operator polynomials in one parameter. We apply the results of [9]

The system has the form

$$
\begin{gathered}
\Delta_{i,} z=\lambda_{i, s} \Delta_{o, i} z \\
\Delta_{k_{1}+k_{2}+\ldots+k_{i-1}+1, i} z_{i}=\lambda_{i, s} \Delta_{o, i} z_{i}
\end{gathered}
$$

Theorem 4. Let the conditions of the theorem1 be fulfilled then operators $\Delta_{o, i}$ have inverses. Moreover, the system(4) has the common eigenvalue if and only if

$$
\operatorname{Ker} \bigcap\left(\Delta_{k_{1}+k_{2}+\ldots+k_{i-1}+1, i}-\lambda_{i, s} \Delta_{o, i}\right) \neq 0 .
$$

## 4. Normally Represented Elementary Operators

Tensor product [10] is a very important technique used in solving problems of norms in Hilbert spaces. Norms are very important properties of operators and interesting studies have been directed on them. In the next paper we give more results on norms.

## References

[1] Atkinson F.V. Multiparameter spectral theory. Bull. Amer. Math. Soc. 1968, 74, 1-27.
[2] Browne P.J. Multiparameter spectral theory. Indiana Univ. Math. J, 24, 3, 1974.
[3] Sleeman B.D. Multiparameter spectral theory in Hilbert space. Pitnam Press, London, 1978, p. 118.
[4] Dzhabarzadeh R.M. Spectral theory of multiparameter system of operators in Hilbert space, Transactions of NAS of Azerbaijan, 1-2, 1999, 33-40.
[5] Dzhabarzadeh R.M, Salmanova G.H. Multtiparameter system of operators, not linearly depending on parameters. American Journal of Mathematics and Mathematical Sciences. 2012, vol.1, No. 2.- p. 39-45.
[6] Dzhabarzadeh R.M. Spectral theory of two parameter s system in finite-dimensional space. Transactions of NAS of Azerbaijan, v. 3-4 1998, p. 12-18.
[7] Balinskii A.I. Generation of notions of Bezutiant and Resultant DAN of Ukr. SSR, ser. ph.-math and tech. of sciences, 1980,2. (in Russian).
[8] Khayniq X. Abstract analog of Resultant for two polynomial bundles Functional analyses and its applications, 1977, 2, p. 94-95.
[9] Dzhabarzadeh R.M. On existence of common eigen value of some operator-bundles, that depends polynomial on parameter. Baku. International Topology conference, 3-9 oct., 1987, Tez. 2, Baku, 1987, p. 93.
[10] John B.P. Tensor products and Multiparameter spectral theory in Hilbert space. Springer Verlag, New York, 2014, p. 17.

