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Certain Subclass of *P*-valent Meromorphic Functions Involving the Extended Multiplier Transformations

R. M. El-Ashwah

Department of Mathematics, Faculty of Science, Damietta University, New Damietta, Egypt

Email address

r_elashwah@yahoo.com

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Abstract

Using the linear operator $I_p^m(\lambda,\ell)(\lambda \geq 0, \ell > 0, p \in \mathrm{N}, m \in \mathrm{N}_0 = \mathrm{N} \cup \{0\})$ for a function $f(z) \in \sum_p$ the class of P-valent meromorphic functions El-Ashwah [6] and the principle of subordination [11], we introduce the class $M_{p,k}^m(\lambda,\ell;\beta;\phi)$, which satisfies the following condition: $\frac{1}{\beta-p}\left[\beta + \frac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{f_{p,k}^m(\lambda,\ell;z)}\right] < \phi(z)$ $(\beta > p; \phi \in p; z \in U)$. Such results as inclusion relationships, integral representations, convolution properties and integral-preserving properties for these functions class are obtained.

Keywords

Subordination, Analytic, Meromorphic, Multivalent, Multiplier Transformations

1. Introduction

Let A_p denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathrm{N} = \{1,2,...\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathrm{C}$ and $|z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows $f \prec g$ in U or $f(z) \prec g(z)\ (z \in U)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in U)$, such that $f(z) = g(w(z))\ (z \in U)$. Indeed it is known that $f(z) \prec g(z)\ (z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function $g(z)$ is univalent in U, then we have the following equivalent (cf., e.g., [11]; see also [12, p.4])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \prec g(U).$$

Let P denote the class of functions of the form:

$$\phi(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic and convex in U and satisfies the following condition

$$\mathrm{Re}\{\phi(z)\} > 0, \quad (z \in U).$$

Also let Σ_p be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathrm{N}), \tag{1.2}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathrm{C}$ and $0 < |z| < 1\} = U \setminus \{0\}$. We note that $\Sigma_1 = \Sigma$ the class of univalent meromophic functions. For functions $f(z) \in \Sigma_p$ given by (1.2) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \quad (p \in \mathrm{N}),$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^{-p}+\sum_{n=1}^{\infty}a_nb_nz^{n-p}=(g*f)(z).$$

El-Ashwah [6] (see also [7-10]) defined a linear operator $I_p^m(\lambda,\ell)(\lambda\geq 0,\ell>0,p\in\mathbb{N},m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ for a function $f(z)\in\Sigma_p$ as follows:

$$I_p^m(\lambda,\ell)f(z)=z^{-p}+\sum_{n=1}^{\infty}\left[\frac{\lambda n+\ell}{\ell}\right]^m a_nz^{n-p}. \tag{1.3}$$

we can write (1.3) as follows:

$$I_p^m(\lambda,\ell)f(z)=(\Phi_{\lambda,\ell}^{p,m}*f)(z),$$

where

$$\Phi_{\lambda,\ell}^{p,m}(z)=z^{-p}+\sum_{n=1}^{\infty}\left[\frac{\lambda n+\ell}{\ell}\right]^m z^{n-p}. \tag{1.4}$$

It is easily verified from (1.3), that

$$\begin{aligned}&\lambda z(I_p^m(\lambda,\ell)f(z))'\\&=\ell I_p^{m+1}(\lambda,\ell)f(z)-(\ell+p\lambda)I_p^m(\lambda,\ell)f(z)\ (\lambda>0).\end{aligned} \tag{1.5}$$

We observe that the operator $I_p^m(\lambda,\ell)$ reduce to several interesting many other operators considered earlier for different choices of λ,ℓ,p and m (see e.g. [1-4], [15-16]).

Throughout this paper, we assume that $p,k\in\mathbb{N},\ell,m\in\mathbb{N}_0,\epsilon_k=\exp(\frac{2\pi i}{k})$ and

$$\begin{aligned}f_{p,k}^m(\lambda,\ell;z)&=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}\ (I_p^m(\lambda,\ell)f(\epsilon_k^j\ z))\\&=z^{-p}+....(f\in\Sigma_p).\end{aligned} \tag{1.6}$$

Clearly, for $k=1$, we have

$$f_{p,1}^m(\lambda,\ell;z)=I_p^m(\lambda,\ell)f(z).$$

Making use of the extended multiplier transformation $I_p^m(\lambda,\ell)$ and the above mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclasses of the class Σ_p of p-valent meromorphic functions.

Definition 1. A function $f(z)\in\Sigma_p$ is said to be in the class $M_{p,k}^m(\lambda,\ell;\beta)$ if it satisfies the following condition:

$$\mathrm{Re}\left(-\frac{z(I_p^m(\lambda,\ell)f(z))'}{f_{p,k}^m(\lambda,\ell;z)}\right)<\beta\quad(\beta>p;z\in U), \tag{1.7}$$

where $f_{p,k}^m(\lambda,\ell;z)\neq 0\ (z\in U^*)$ is defined by (1.6).

Also, a function $f(z)\in\Sigma_p$ is said to be in the class $N_{p,k}^m(\lambda,\ell;\beta)$ if and only if

$$-\frac{zf'(z)}{p}\in M_{p,k}^m(\lambda,\ell;\beta).$$

Remark 1. (i)Putting $\lambda=k=1$ and $m=\ell=0$ in the classes $M_{p,k}^m(\lambda,\ell;\beta)$ and $N_{p,k}^m(\lambda,\ell;\beta)$ we obtain the function classes $M_p(\beta)$ and $N_p(\beta)$ which are introduced and studied by Wang et al. [17] and Wang et al. [18];

(ii) Putting $p=\lambda=k=1$ and $m=\ell=0$ in the class $M_{p,k}^m(\lambda,\ell;\beta)$ we obtain the function class $\Sigma_N^{**}(\beta)$ which are introduced and studied by Sarangi and Uralegaddi [14].

Definition 2. A function $f(z)\in\Sigma_p$ is said to be in the class $M_{p,k}^m(\lambda,\ell;\beta;\phi)$ if it satisfies the following subordination condition:

$$\frac{1}{\beta-p}\left(\beta+\frac{z(I_p^m(\lambda,\ell)f(z))'}{f_{p,k}^m(\lambda,\ell;z)}\right)\prec\phi(z)$$
$$(\beta>p;\phi\in P;z\in U), \tag{1.8}$$

where $f_{p,k}^m(\lambda,\ell;z)\neq 0\ (z\in U)$ is defined by (1.6).

In this paper, we aim and proving such results as inclusion relationships, integral representations, convolution properties and integral-preserving properties for the function class $M_{p,k}^m(\lambda,\ell;\beta;\phi)$.

2. Preliminaries

In order to establish our main results, we shall use of the following lemmas.

Lemma 1 [5, 11].Let $\beta,\gamma\in\mathbb{C}$. Suppose also that $\phi(z)$ is convex and univalent in U with

$\phi(0)=1$ and $\mathrm{Re}\{\beta\phi(z)+\gamma\}>0\quad(z\in U)$.

If $p(z)$ is analytic in U with $p(0)=1$, then the following subordination:

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\phi(z)$$

implies that

$$p(z)\prec\phi(z)\ \ (z\in U).$$

Lemma 2 [13]. Let $\beta,\gamma\in\mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in U with

$$\phi(0)=1 \text{ and } \mathrm{Re}\{\beta\phi(z)+\gamma\}>0\quad(z\in U).$$

Also let

$$q(z)\prec\phi(z).$$

If $p(z)\in P$ and satisfies the following subordination:

$$p(z)+\frac{zp'(z)}{\beta q(z)+\gamma}\prec\phi(z),$$

then

$$p(z)\prec\phi(z).$$

Lemma 3. Let $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. Then

$$\frac{1}{\beta-p}\left(\beta+\frac{z(f_{p,k}^{m}(\lambda,\ell;z))'}{f_{p,k}^{m}(\lambda,\ell;z)}\right)\prec\phi(z). \tag{2.1}$$

Proof. In view of (1.6), we replace z by $\in_k^j z$ $(j=0,1,2,..,k-1)$ in $f_{p,k}^{m}(\lambda,\ell;z)$. We thus obtain

$$f_{p,k}^{m}(\lambda,\ell;\in_k^j z)=\frac{1}{k}\sum_{n=0}^{k-1}\in_k^{np}\ (I_p^m(\lambda,\ell)f)(\in_k^{n+j} z)$$

$$=\in_k^{-jp}\ \frac{1}{k}\sum_{n=0}^{k-1}\in_k^{(n+j)p}\ (I_p^m(\lambda,\ell)f)(\in_k^{n+j} z)$$

$$=\in_k^{-jp}\ f_{p,k}^{m}(\lambda,\ell;z). \tag{2.2}$$

Differentiating both sides of (1.6) with respect to z, we obtain

$$(f_{p,k}^{m}(\lambda,\ell;z))'=\frac{1}{k}\sum_{j=0}^{k-1}\in_k^{j(p+1)}\ (I_p^m(\lambda,\ell)f)'(\in_k^j z). \tag{2.3}$$

Therefore, from (2.2) and (2.3), we find that

$$\frac{1}{\beta-p}\left(\beta+\frac{z(f_{p,k}^{m}(\lambda,\ell;z))'}{f_{p,k}^{m}(\lambda,\ell;z)}\right)$$

$$=\frac{1}{\beta-p}\left(\beta+\frac{1}{k}\sum_{j=0}^{k-1}\frac{\in_k^{j(p+1)}\ z(I_p^m(\lambda,\ell)f)'(\in_k^j z)}{f_{p,k}^{m}(\lambda,\ell;z)}\right)$$

$$=\frac{1}{\beta-p}\left(\beta+\frac{1}{k}\sum_{j=0}^{k-1}\frac{\in_k^{j}\ z(I_p^m(\lambda,\ell)f)'(\in_k^j z)}{f_{p,k}^{m}(\lambda,\ell;\in_k^j z)}\right). \tag{2.4}$$

Moreover, since $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$, it follows that

$$\frac{1}{\beta-p}\left(\beta+\frac{\in_k^{j}\ z(I_p^m(\lambda,\ell)f)'(\in_k^j z)}{f_{p,k}^{m}(\lambda,\ell;\in_k^j z)}\right)\prec\phi(z) \tag{2.5}$$
$$(j=0,1,..,k-1;z\in U).$$

Finally, by noting that $\phi(z)$ is convex and univalent in U, from (2.4) and (2.5), we conclude that the assertion (2.1) of Lemma 3 holds true.

3. Properties of the Function Class $M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$

In this section, we obtain some inclusion relationships for the function class $M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$.

Unless otherwise mentioned we shall assume throughout the paper that $\lambda>0,\ell\geq 0,\beta>p,p,k\in\mathbb{N}$ and $m\in\mathbb{N}_0$.

Theorem 1. Let $\phi\in P$ with

$$\operatorname{Re}\left\{(\beta-p)\phi(z)-\beta+p+\frac{\ell}{\lambda}\right\}>0\ \ (z\in U),$$

then

$$M_{p,k}^{m+1}(\lambda,\ell;\beta;\beta;\phi)\subset M_{p,k}^{m}(\lambda,\ell;\beta;\phi).$$

Proof. Making use of the relationships in equations (1.5) and (1.6), we know that

$$z\left(f_{p,k}^{m}(\lambda,\ell;\phi)\right)'+\left(p+\frac{\ell}{\lambda}\right)f_{p,k}^{m}(\lambda,\ell;z)$$

$$=\frac{\left(\frac{\ell}{\lambda}\right)}{k}\sum_{j=0}^{k-1}\in_k^{jp}\ \left(I_p^{m+1}(\lambda,\ell)f(\in_k^j z)\right)=\frac{\ell}{\lambda}f_{p,k}^{m+1}(\lambda,\ell;z) \tag{3.1}$$

Let $f\in M_{p,k}^{m+1}(\lambda,\ell;\beta;\phi)$ and suppose that

$$(\beta-p)p(z)-\beta=\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}\quad(z\in U). \tag{3.2}$$

Then $p(z)$ is analytic in U and $p(0)=1$. It follows from (3.1) and (3.2) that

$$(\beta-p)p(z)-\beta+p+\frac{\ell}{\lambda}=\frac{\ell}{\lambda}\frac{f_{p,k}^{m+1}(\lambda,\ell;z)}{f_{p,k}^{m}(\lambda,\ell;z)}. \tag{3.3}$$

Differentiating both sides of (3.3) logarithmically with respect to z and using (3.2), we obtain

$$p(z)+\frac{zp'(z)}{(\beta-p)p(z)-\beta+p+\frac{\ell}{\lambda}}$$
$$=\frac{1}{(\beta-p)}\left(\beta+\frac{z\left(f_{p,k}^{m+1}(\lambda,\ell;z)\right)'}{f_{p,k}^{m+1}(\lambda,\ell;z)}\right). \tag{3.4}$$

From (3.4) and Lemma 3 (with m replaced by ($m+1$)), we can see that

$$p(z)+\frac{zp'(z)}{(\beta-p)p(z)-\beta+p+\frac{\ell}{\lambda}}\prec\phi(z)\quad(z\in U). \tag{3.5}$$

Since $\operatorname{Re}\left\{(\beta-p)\phi(z)-\beta+p+\frac{\ell}{\lambda}\right\}>0\ (z\in U)$, by Lemma 1, we have

$$p(z)=\frac{1}{(\beta-p)}\left(\beta+\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}\right)\prec\phi(z)\ \ (z\in U). \quad (3.6)$$

By setting

$$q(z)=\frac{1}{(\beta-p)}\left(\beta+\frac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}\right)\quad (z\in U), \quad (3.7)$$

we observe that $q(z)$ is analytic in U and $q(0)=1$. It follows from (1.5) and (3.7) that

$$\begin{aligned}&\left((\beta-p)q(z)-\beta\right)f_{p,k}^{m}(\lambda,\ell;z)\\&=\frac{\ell}{\lambda}I_p^{m+1}(\lambda,\ell)f(z)-\left(p+\frac{\ell}{\lambda}\right)I_p^m(\lambda,\ell)f(z).\end{aligned} \quad (3.8)$$

Differentiating both sides of (3.8) with respect to z and using (3.7), we obtain

$$\begin{aligned}&(\beta-p)zq'(z)+\left(p+\frac{\ell}{\lambda}+(\beta-p)\,p(z)-\beta\right)((\beta-p)q(z)-\beta)\\&=\frac{\ell}{\lambda}\cdot\frac{z\left(I_p^{m+1}(\lambda,\ell)f(z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}.\end{aligned} \quad (3.9)$$

From (3.2), (3.3) and (3.9), we can obtain

$$\begin{aligned}&q(z)+\frac{zq'(z)}{(\beta-p)p(z)-\beta+p+\frac{\ell}{\lambda}}\\&=\frac{1}{(\beta-p)}\left(\beta+\frac{z\left(I_p^{m+1}(\lambda,\ell)f(z)\right)'}{f_{p,k}^{m+1}(\lambda,\ell;z)}\right)\prec\phi(z)\quad(z\in U).\end{aligned}$$

Since

$$p(z)\prec\phi(z)\quad(z\in U)$$

and

$$\operatorname{Re}\left\{(\beta-p)\phi(z)-\beta+p+\frac{\ell}{\lambda}\right\}>0\ \ (z\in U),$$

it follows from (3.9) and Lemma 2 that

$$q(z)\prec\phi(z)\quad(z\in U),$$

that is, that $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. This implies that

$$M_{p,k}^{m+1}(\lambda,\ell;\beta;\phi)\subset M_{p,k}^{m}(\lambda,\ell;\beta;\phi).$$

Hence the proof of Theorem 1 is completed.

4. Integral Representation

In this section, we obtain a number of integral representations associated with the function class $M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$.

Theorem 2. Let $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. Then

$$f_{p,k}^{m}(\lambda,\ell;z)=z^{-p}\exp\left\{\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\phi(w(\in_k^j\xi))-1}{\xi}d\xi\right\}, \quad (4.1)$$

where $f_{p,k}^{m}(\lambda,\ell;z)$ is defined by (1.6), $w(z)$ is analytic in U and satisfy $w(0)=1$ and $|w(z)|<1\ (z\in U)$.

Proof. Suppose that $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. Then condition (1.8) can be written as follows:

$$\frac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}=(\beta-p)\phi(w(z))-\beta\ (z\in U), \quad (4.2)$$

where $w(z)$ is analytic in U and satisfy $w(0)=1$ and $|w(z)|<1\ (z\in U)$. Replacing z by $\in_k^j z\ (j=0,1,...,k-1)$ in (4.2), we observe that (4.2) becomes

$$\begin{aligned}&\frac{\in_k^j z\left(I_p^m(\lambda,\ell)f(\in_k^j z)\right)'}{f_{p,k}^{m}(\lambda,\ell;\in_k^j z)}\\&=(\beta-p)(\phi(w(\in_k^j z)))-\beta\qquad(z\in U).\end{aligned} \quad (4.3)$$

We note that

$$f_{p,k}^{m}(\lambda,\ell;\in_k^j z)=\in_k^{-jp}f_{p,k}^{m}(\lambda,\ell;z)\ (z\in U).$$

Thus, by letting $j=0,1,....,k-1$ in (4.3), successively, and summing the resulting equations, we have

$$\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}=\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\phi(w(\in_k^j z))-\beta\ \ (z\in U). \quad (4.4)$$

From (4.4), we get

$$\frac{\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}+\frac{p}{z}=\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\left[\frac{\phi(w(\in_k^j z))-1}{z}\right](z\in U), \quad (4.5)$$

which, upon integration, yields

$$\log\left(z^p f_{p,k}^{m}(\lambda,\ell;z)\right)=\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\phi(w(\in_k^j\xi))-1}{\xi}d\xi. \quad (4.6)$$

Then, the assertion (4.1) of Theorem 2 can now easily obtained from (4.6).

Remark 2. Putting $k=1, m=0$ and $\phi(z)=\frac{1+z}{1-z}$ in Theorem 2 we obtain the result obtained by Wang et al. [17, Th. 2].

Theorem 3. Let $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. *Then*

$$I_p^m(\lambda,\ell)f(z) = \int_0^z \frac{(\beta-p)\phi(w(\zeta))-\beta}{\zeta^{p+1}}.\exp\left(\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\phi(w(\in_k^j \xi))-1}{\xi}d\xi\right)d\zeta, \quad (4.7)$$

where $w(z)$ is analytic in U and satisfy $w(0)=1$ and $|w(z)|<1\ (z\in U)$.

Proof. Suppose that $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. Then, from (4.1) and (4.2), we have

$$\left(I_p^m(\lambda,\ell)f(z)\right)' = \frac{f_{p,k}^m(\lambda,\ell;z)}{z}\left((\beta-p)\phi(w(z))-\beta\right)$$

$$= \frac{((\beta-p)\phi(w(z))-\beta)}{z^{p+1}}.\exp\left(\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\phi(w(\in_k^j \xi))-1}{\xi}d\xi\right), \quad (4.8)$$

which, upon integration, leads us easily to the assertion (4.7) of Theorem 3.

Theorem 4. Let $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. *Then*

$$I_p^m(\lambda,\ell)f(z) = \int_0^z \frac{(\beta-p)\phi(w_2(\zeta))-\beta}{\zeta^{p+1}}.\exp\left(\int_0^{\xi}\frac{(\beta-p)[\phi(w_1(\xi))-1]}{\xi}d\xi\right)d\zeta, \quad (4.9)$$

where $w_i(z)(i=1,2)$ are analytic in U with $w_i(0)=0$ and $|w_i(z)|<1 (z\in U; i=1,2)$.

Proof. Suppose that $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. We then find from (2.1) that

$$\frac{z\left(f_{p,k}^m(\lambda,\ell;z)\right)'}{f_{p,k}^m(\lambda,\ell;z)} = (\beta-p)\phi(w_1(z))-\beta \quad (z\in U), \quad (4.10)$$

Where $w_1(z)$ is analytic in U with $w_1(0)=1$. Thus, by similarly applying the method of proof of Theorem 3, we find that

$$f_{p,k}^m(\lambda,\ell;z) = z^{-p}.\exp\left(\int_0^z \frac{(\beta-p)[\phi(w_1(\xi))-1]}{\xi}d\xi\right). \quad (4.11)$$

It now follows from (4.2) and (4.11) that

$$(I_p^m(\lambda,\ell)f(z))' = \frac{f_{p,k}^m(\lambda,\ell;z)}{z}((\beta-p)\phi(w_2(z))-\beta)$$

$$= \frac{(\beta-p)\phi(w_2(z))-\beta}{z^{p+1}}.\exp\left(\int_0^z \frac{(\beta-p)[\phi(w_1(\xi))-1]}{\xi}d\xi\right),$$

(4.12)

where $w_i(z)(i=1,2)$ are analytic in U with $w_i(0)=0$ and $|w_i(z)|<1 (z\in U; i=1,2)$. Integrating both sides of (4.12), we will obtain the assertion (4.9) of Theorem 4.

5. Convolution Properties

In this section, we derive some convolution properties for the class $M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$.

Theorem 5. Let $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. *Then*

$$f(z) = \left[\int_0^z \frac{(\beta-p)\phi(w(\zeta))-\beta}{\zeta^{p+1}}.\exp\left(\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\phi(w(\in_k^j \xi))-1}{\xi}d\xi\right)d\zeta\right]*$$

$$*\left(\sum_{n=0}^{\infty}\left(\frac{\ell}{\ell+\lambda n}\right)^m z^{n-p}\right), \quad (5.1)$$

where $w(z)$ is analytic in U with $w(0)=1$ and $|w(z)|<1\ (z\in U)$.

Proof. In view of (1.3) and (4.7), we know that

$$\int_0^z \frac{(\beta-p)\phi(w(\zeta))-\beta}{\zeta^{p+1}}.\exp\left(\frac{(\beta-p)}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\phi(w(\in_k^j \xi))-1}{\xi}d\xi\right)d\zeta$$

$$=\left(\sum_{n=0}^{\infty}\left(\frac{\ell+\lambda n}{\ell}\right)^m z^{n-p}\right)*f(z) = \Phi_{p,\lambda,\ell}^m(z)*f(z), \quad (5.2)$$

where $\Phi_{p,\lambda,\ell}^m(z)$ is given by (1.4).

Thus, from (5.2), we can easily get the assertion (5.1) of Theorem 5.

Theorem 6. Let $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. *Then*

$$f(z) = \left[\int_0^z \frac{(\beta-p)\phi(w_2(\zeta))-\beta}{\zeta^{p+1}}.\exp\left(\int_0^{\zeta}\frac{(\beta-p)[\phi(w_1(\xi))-1]}{\xi}d\xi\right)d\zeta\right]*$$

$$*\left(\sum_{n=0}^{\infty}\left(\frac{\ell}{\ell+\lambda n}\right)^m z^{n-p}\right), \quad (5.3)$$

where $w_j(z)(j=1,2)$ are analytic in U with $w_j(0)=0$ and $|w_j(z)|<1 (z\in U; j=1,2)$.

Proof. In view of (1.4) and (4.9), we know that

$$\int_0^z \frac{(\beta-p)\phi(w_2(\zeta))-\beta}{\zeta^{p+1}}.\exp\left(\int_0^{\zeta}\frac{(\beta-p)[\phi(w_1(\xi))-1}{\xi}d\xi\right)d\zeta$$

$$=\left(\sum_{n=0}^{\infty}\left(\frac{\ell+\lambda n}{\ell}\right)^m z^{n-p}\right)*f(z) = \Phi_{p,\lambda,\ell}^m(z)*f(z). \quad (5.4)$$

Thus, from (5.4), we easily obtain (5.3).

Theorem 7. Let $f\in\Sigma_p$ and $\phi\in P$. Then $f \in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$ if and only if

$$z^p\left\{f*\left[\left(\sum_{n=0}^{\infty}\left(\frac{\ell+\lambda n}{\ell}\right)^m(n-p)z^{n-p}\right)\right.\right.$$

$$\left.\left.-\left((\beta-p)\phi(e^{i\theta})-\beta\right)\left(\sum_{n=0}^{\infty}\left(\frac{\ell+\lambda n}{\ell}\right)^m z^{n-p}\right)*\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z^{-p}}{1-\epsilon^{\nu}z}\right)\right]\right\}\neq 0$$

$$(z\in U;\ 0\leq\theta<2\pi). \tag{5.5}$$

Proof. Suppose that $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. Since

$$\frac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}\prec(\beta-p)\phi(z)-\beta$$

is equivalent to

$$\frac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}\neq(\beta-p)\phi(e^{i\theta})-\beta\ (z\in U;0\leq\theta<2\pi), \tag{5.6}$$

it is easy to see that the condition (5.6) can be written as follows:

$$z^p\left[z\left(I_p^m(\lambda,\ell)f(z)\right)'-f_{p,k}^{m}(\lambda,\ell;z)\left((\beta-p)\phi(e^{i\theta})-\beta\right)\right]\neq 0 \tag{5.7}$$
$$(z\in U;0\leq\theta<2\pi).$$

On the other hand, we know from (1.3) that

$$z\left(I_p^m(\lambda,\ell)f(z)\right)'=\left(\sum_{n=0}^{\infty}\left(\frac{\ell+\lambda n}{\ell}\right)^m(n-p)z^{n-p}\right)*f(z). \tag{5.8}$$

Also, from the definition of $f_{p,k}^{m}(\lambda,\ell;z)$, we have

$$f_{p,k}^{m}(\lambda,\ell;z)=I_p^m(\lambda,\ell)f(z)*\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z^{p}}{1-\epsilon^{\nu}z}\right)$$

$$=\left(\sum_{n=0}^{\infty}\left(\frac{\ell+\lambda n}{\ell}\right)^m z^{n-p}\right)*\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z^{-p}}{1-\epsilon^{\nu}z}\right)*f(z). \tag{5.9}$$

Upon substituting from (5.8) and (5.9) in (5.7), we can easily obtain the convolution property (5.5) asserted by Theorem 7.

Remark 3. Putting $k=1,m=0$ and $\phi(z)=\frac{1+e^{i\theta}}{1-e^{i\theta}}(0\leq\theta<2\pi)$ in Theorem 7 we obtain the result obtained by Wang et al. [17, Th. 3].

6. Integral-Preserving Properties

In this section, we prove some integral-preserving properties for the class $M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$.

Theorem 8. Let $\phi\in P$ *and*

$$\mathrm{Re}\left\{(\beta-p)\phi(z)-\beta+p+\mu\right\}>0\ \ (z\in U).$$

If $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$, then the function $F(z)\in\Sigma_p$ defined by

$$F(z)=\frac{\mu}{z^{\mu+p}}\int_0^z t^{\mu+p-1}f(t)dt\ \ (\mu>0;z\in U) \tag{6.1}$$

belongs to the class $M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$.

Proof. Let $f\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. Then, from (6.1), we find that

$$\begin{aligned}&z\left(I_p^m(\lambda,\ell)F(z)\right)'+(\mu+p)I_p^m(\lambda,\ell)F(z)\\&=\mu I_p^m(\lambda,\ell)f(z).\end{aligned} \tag{6.2}$$

Thus, in view of (1.6) and (6.1), we have

$$\begin{aligned}&z\left(F_{p,k}^{m}(\lambda,\ell;z)\right)'+(\mu+p)F_{p,k}^{m}(\lambda,\ell;z)\\&=\mu f_{p,k}^{m}(\lambda,\ell;z)\end{aligned} \tag{6.3}$$

We now put

$$H(z)=\frac{1}{\beta-p}\left(\beta+\frac{z\left(F_{p,k}^{m}(\lambda,\ell;z)\right)'}{F_{p,k}^{m}(\lambda,\ell;z)}\right)\ \ (z\in U). \tag{6.4}$$

Then $H(z)$ is analytic in U and $H(0)=1$. It follows from (6.3) and (6.4) that

$$(\beta-p)H(z)-\beta+p+\mu=\mu\frac{f_{p,k}^{m}(\lambda,\ell;z)}{F_{p,k}^{m}(\lambda,\ell;z)}. \tag{6.5}$$

Differentiating both sides of (6.5) logarithmically with respect to z and using Lemma 3, we obtain

$$\begin{aligned}&H(z)+\frac{zH'(z)}{(\beta-p)H(z)-\beta+p+\mu}\\&=\frac{1}{\beta-p}\left(\beta+\frac{z(f_{p,k}^{m}(\lambda,\ell;z))'}{f_{p,k}^{m}(\lambda,\ell;z)}\right)\prec\phi(z).\end{aligned} \tag{6.6}$$

Since $\mathrm{Re}\left\{(\beta-p)\phi(z)-\beta+p+\mu\right\}>0\ (z\in U)$, it follows from (6.6) and Lemma 1 that $H(z)\prec\phi(z)\ (z\in U)$. Furthermore, we suppose that

$$G(z)=\frac{1}{\beta-p}\left(\beta+\frac{z(I_p(\lambda,\ell)F(z))'}{F_{p,k}^{m}(\lambda,\ell;z)}\right)\quad(z\in U).$$

The remainder of the proof of Theorem 8 is similar to that of Theorem 1. We, therefore, choose to omit the analogous details involved. We thus find that

$$G(z)\prec\phi(z),$$

which implies that $F(z)\in M_{p,k}^{m}(\lambda,\ell;\beta;\phi)$. This completes the proof of Theorem 8.

Remark 4. By specializing the parameters λ, ℓ, p and m, we can obtain corresponding results for various subclasses associated with various operators.

7. Conclusion

The author used the operator $I_p^m(\lambda,\ell)$ to define the class $M_{p,k}^m(\lambda,\ell;\beta;\phi)$ of meromorphic analytic functions. Inclusion relationships, integral representations, convolution properties and integral-preserving properties for these function class are obtained. Some results concerning to the class $N_{p,k}^m(\lambda,\ell;\beta;\phi)$ can be obtained from the relation $f(z)\in N_{p,k}^m(\lambda,\ell;\beta)$ if and only if $-\frac{zf'(z)}{p}\in M_{p,k}^m(\lambda,\ell;\beta)$.

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